SOME STATIC CYLINDRICALLY SYMMETRIC PERFECT FLUIDS WITH NON-ZERO SPIN

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Abstract- In this paper taking a static cylindrically symmetric perfect fluid distribution in Einstein-Cartan theory we have solved the field equations using a suitable equation of state and by choosing a specific form for one of the metric potentials. We have assumed the spins of all the individual particles composing the fluid to be aligned along the symmetric axis. Pressure and density have been calculated and the constants appearing in the solution have been evaluated using boundary conditions.

Keywords- Einstein-Cartan theory, Cylindrically Symmetric, Perfect Fluid Distribution, Spin density, Stellar Bodies.

1. INTRODUCTION

The discovery of Pulsars which gave a big impetus to the developments in relativistic astrophysics, particularly in the theory of neutron stars. Since the rotating neutron star model for a pulsar seems to be almost undisputed (Borner [1]), it is now essential to study the possible interval structure of neutron stars with respect to gravitation and the associated geometry of space time therein. It is seen through observations that a very high magnetic field is associated with them in comparison with any other compact objects in the universe. It is not unlikely that this magnetic field might induce a spin polarization of the neutrons composing the fluid of a neutron star. In 1973, Kerlick [4] studied that the pure torsion effects in neutron star is unobservable compared to those of magnetic dipole interactions. It is probable in the case when spins are aligned that there would be a substantial non zero spin density which would then play along with the mass density, a dynamical role in influencing the geometry of space time containing the fluid. One can study such problems in the framework of Einstein-Cartan theory.

Prasanna [13] has considered a study of static fluid spheres in E-C Theory as an attempt to investigate whether Einstein-Cartan theory admits self gravitating fluid systems. Static fluid spheres in Einstein-Cartan theory have been also studied by Kerlick [5], Kuchowicz [6-9], Skinner and Webb [16] and Singh and Yadav [15], whereas Yadav and Prasad [19] have studied non static perfect fluid spheres with conformal flatness. However, since in spherical symmetry it is assumed that spins are all aligned in radial direction (implying the presence of a magnetic monopole at the centre) the picture is not very physical. Further, as a rotating system cannot be spherical, naturally it seems desirable to study axi-symmetric distributions which are more physical. Keeping this fact in mind Prasanna [14] and Yadav et. al. [20] have considered the simplest axi-symmetric system namely a static cylinder of perfect fluid in Einstein- Cartan Theory.

Here in this paper taking a static cylindrically symmetric perfect fluid distribution in Einstein-Cartan theory we have solved the field equations using a suitable equation of state and by choosing a specific form for one of the metric potentials. We have assumed the spins of all the individual particles composing the fluid to be aligned along the symmetric axis. Pressure and density have been calculated and the constants appearing in the solution have been evaluated using boundary conditions.

2. METRIC AND THE CURVATURE

Here we take the ststic cylindrically symmetric metric given by

(2.1) $$ds^2 = -e^{2\alpha-2\beta}\left(dr^2+dz^2\right)-r^2e^{-2\beta}d\phi^2+e^{2\beta}dt^2$$

Where α & β are functions of r alone.

Following Trautman [17] we introduce a set of coframes θ^i and the torsion 2 from $\oplus^i$ defined as

(2.2) $$\oplus^i = d\theta^i + w^i_k$$

$$= \frac{1}{2}Q^i_{jk}\theta^j \wedge \theta^k$$

Where w^i_j are connection 1-forms such that

(2.3) $$w^i_k = \Gamma^i_{jk}\theta^j$$

and Q^i_{jk} is the torsion tensor.

The curvature two from Ω^i_j is defined as

(2.4) $$\Omega^i_j = dw^i_j + w^i_k \wedge w^k_j$$

$$= \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l$$

and

(2.5) $$Q^i_{jk} - \delta^i_j Q^l_{lk} - \delta^i_k Q^l_{jl} = -kS^j_{jk}$$

where R^l_{jkl} is the curvature tensor and S^i_{jk} is the spin tensor.

The classical description of spin is defined by the relation

(2.6) $$S^i_{jk} = u^i S_{jk} \text{ with } S_{jk}u^k = 0$$

Where u^i is the velocity for vector and S_{ij} is the intrinsic angular momentum tensor.

For the metric (4.2.1) we have the orthonormal tetrad

(2.7) $$\theta^1 = e^{\alpha-\beta}dr,\ \theta^2 = re^{-\beta}d\phi,\ \theta^3 = e^{\alpha-\beta}dz,\ \theta^4 = e^{\beta}dt$$

The metric (4.2.1) now become

(2.8) $$ds^2 = -\{(\theta^1)^2 + (\theta^2)^2 + (\theta^3)^2 - (\theta^4)^2\}$$

So that

$$g_{ij} = -diag(-1,-1,-1,1)$$

We suppose that the spins of the individual particles composing the fluid are all aligned along the symmetric (z-axis).

Therefore, the only non zero components of the spin tensor S_{ij} are

(2.9) $$S_{12} = -S_{21} = K(say)$$

Also since the fluid distributions is static the velocity for vector $u^i = \delta^i_4$ and hence the non-zero components of S^i_{jk} are

(2.10) $$S^4_{12} = -S^4_{21} = K$$

Therefore, from the Cartan equation (4.2.5), we have the non zero components of torsion tensor Q^i_{jk} to be

(2.11) $$Q^4_{12} = -Q^4_{21} = -kK$$

the others are zero.

Using (4.2.11) in (4.2.2) we get the torsion 2 - form $\oplus^i$ to be

(2.12) $$\oplus^1 = 0,\ \oplus^2 = 0,\ \oplus^3 = 0,\ \oplus^4 = -kK(\theta^1 \wedge \theta^2)$$

The components of connection 1 – form w^i_j are found to be

(2.13) $$w^1_2 = -w^2_1 = \frac{e^{\beta-\alpha}}{\Gamma}(1-\Gamma\beta^1)\theta^1 + \frac{1}{2}kk\theta^4$$

$$w^1_4 = w^4_1 = e^{\beta-\alpha}\beta^1\theta^4 + \frac{1}{2}kK\theta^2$$

$$w^2_4 = w^4_2 = -\frac{1}{2}kK\theta^1$$

$$w^2_3 = -w^3_2 = 0$$

$$w^3_4 = w^4_3 = 0$$

The non zero components of curvature form Ω^i_j are obtained as

(2.14)
$$\Omega^1_2 = -\Omega^2_1 = \left[e^{2(\beta-\alpha)}\left(\beta'' + \frac{\alpha_1}{r} - \alpha'\beta' + \frac{\beta'}{r}\right) + \frac{1}{4}k^2K^2 \right]$$
$$(\theta^1 \wedge \theta^2) + \frac{1}{2}k\left[e^{\beta-\alpha}\left(K + 2K\beta^1\right)\right](\theta^1 \wedge \theta^4)$$

$$\Omega^1_3 = -\Omega^3_1\left[e^{2(\beta-\alpha)}(\beta'' - \alpha'')\right](\theta^1 \wedge \theta^3)$$

$$\Omega^2_3 = -\Omega^3_2 = \left[e^{2(\beta-\alpha)}\left(\beta - \frac{1}{r}\right)(\alpha' - \beta')\right](\theta^2 \wedge \theta^3)$$

$$+\left[\frac{1}{2}kKe^{\beta-\alpha}(\alpha - \beta)\right](\theta^4 \wedge \theta^3)$$

$$\Omega^1_4 = \Omega^4_1 = \left[e^{2(\beta-\alpha)}\left(\beta'' + \beta'^2 - \beta'\alpha'\right) + \frac{1}{4}k^2K^2 \right]$$

$$(\theta^1 \wedge \theta^4) + \frac{1}{2}ke^{\beta-\alpha}K(\theta^1 \wedge \theta^2)$$

$$\Omega^2_4 = -\Omega^4_2 = \left[e^{2(\beta-\alpha)}\beta'\left(\frac{1}{r} - \beta'\right) + \frac{1}{4}k^2K^2 \right](\theta^2 \wedge \theta^4)$$

$$\Omega^3_4 = \Omega^4_3 = \left[e^{2(\beta-\alpha)}\beta(\beta - \alpha')\right](\theta^4 \wedge \theta^3)$$

$$-\left[\frac{1}{2}kKe^{\beta-\alpha}(\alpha - \beta')\right](\theta^2 \wedge \theta^3)$$

Comparison of these results with

$$\Omega^i_j = \frac{1}{2}R^i_{jkl}\theta^k \wedge \theta^l$$

Immediately yields the following components of the Reimann tensor R^i_{jkl} as

(2.15)
$$R^1_{212} = e^{2(\beta-\alpha)}\left(\beta'' - \beta'\alpha + \frac{\alpha' + \beta'}{r}\right) + \frac{1}{4}k^2K^2$$

$$R^1_{214} = \frac{1}{2}ke^{\beta-\alpha}\left(K' + 2k\beta'\right)$$

$$R^1_{313} = e^{2(\beta-\alpha)}(\beta''-\alpha'')$$

$$R^2_{323} = e^{2(\beta-\alpha)}\left(\beta^1 - \frac{1}{r}\right)(\alpha^1 - \beta^1)$$

$$R^2_{343} = \frac{1}{2}kKe^{\beta-\alpha}(\alpha-\beta)$$

$$R^4_{112} = \frac{1}{2}ke^{\beta-\alpha}K^1$$

$$R^4_{114} = e^{2(\beta-\alpha)}(\beta''+2\beta'^2-\beta'\alpha)+\frac{1}{4}k^2K^2$$

$$R^4_{224} = e^{2(\beta-\alpha)}\left(\frac{\beta}{r}-\beta'^2\right)+\frac{1}{4}k^2K^2$$

$$R^4_{343} = e^{2(\beta-\alpha)}\beta'(\beta'-\alpha')$$

$$R^4_{323} = -\frac{1}{2}kKe^{\beta-\alpha}(\alpha'-\beta')$$

3. THE FIELD EQUATIONS

Considering the perfect fluid material distribution with anisotropic pressure, the symmetric energy momentum tensor $\overline{T}^i_j$ is given by

(3.1) $$\overline{T}^i_j = diag\{-p_r, -p_\phi, -p_z, \rho\}$$

The Einestein-Cartan field equations are

(3.2) $$G^i_j \equiv R^i_j - \frac{1}{2}R\delta^i_j = -kt^i_j$$

(3.3) $$Q^i_{jk} - \delta^i_j Q^l_{lk} - \delta^i_k Q^l_{jl} = -kS^i_{jk}$$

Where R_{ij} is Ricci tensor, R is scalar of curvature and t^i_j is canonical asymmetric energy momentum tensor.

The canonical asymmetric energy momentum tensor is given by

(3.4) $$t^i_j = \overline{T}^i_j + \frac{1}{2}g^{im}\square_k S^k_{jm}$$

The non zero components of the canonical tensor t^i_j are

(3.5) $$t^1_1 = \overline{T}^1_1 = -p_r$$

$$t^2_2 = \overline{T}^2_2 = -p_\phi$$

$$t_3^3 = \overline{T}_3^3 = -p_z$$

$$t_4^4 = \overline{T}_4^4 = \rho$$

$$t_2^4 = \frac{1}{2}Ke^{\beta-\alpha}\beta'$$

$$t_4^2 = \frac{1}{2}Ke^{\beta-\alpha}\beta'$$

Using equations (3.2) and (3.5) the field equation may be written as (Prasanna [14])

(3.6) $$e^{2(\beta-\alpha)}\left(2\beta''-\alpha''+\frac{2\beta'}{r}-\beta'^2\right)+\frac{1}{4}k^2K^2=-k\rho$$

(3.7) $$e^{2(\beta-\alpha)}\left(\beta'^2-\frac{\alpha'}{r}\right)-\frac{1}{4}k^2K^2=-kp_r$$

(3.8) $$e^{2(\beta-\alpha)}\left(-\alpha'-\beta'^2\right)-\frac{1}{4}k^2K^2=kp_\phi$$

(3.9) $$e^{2(\beta-\alpha)}\left(-\beta'^2+\frac{\alpha'}{r}\right)-\frac{1}{4}k^2K^2=kp_z$$

(3.10) $$e^{2(\beta-\alpha)}\left(K'+K\alpha'-K\beta'\right)=-ke^{\beta-\alpha}\beta$$

(3.11) $$e^{(\beta-\alpha)}\left(K'+K\alpha'+K\beta'\right)=ke^{\beta-\alpha}\beta'$$

Adding (3.10) and (3.11), we get

(3.12) $$K'+K\alpha'=0$$

Which on integration gives

(3.13) $$K=He^{-\alpha}$$

Where H is an arbitrary constant to be determined. The conservation equation for J = 1 gives the continuity equation.

(3.14) $$\frac{dp_r}{dr}+(\rho+p_r)^{\beta'}-(p_r-p_\phi)\left(\beta'-\frac{1}{r}\right)$$

$$-(\beta'-\alpha')(p_r-p_z)=-\frac{1}{2}kK(K'+k\beta')$$

Setting $k=-\frac{8\pi G}{c^2}$ with G = l, c = l

We can write the field equation as

(3.15) $$8\pi p=16\pi^2K^2+e^{2(\beta-\alpha)}\left(2\beta''-\alpha''+\frac{2\beta'}{r}-\beta'^2\right)$$

(3.16) $$8\pi p_r = -8\pi p_z = 16\pi^2 K^2 + e^{2(\beta-\alpha)}\left(\frac{\alpha'}{r} - \beta'^2\right)$$

(3.17) $$8\pi p_\phi = 16\pi^2 K^2 + e^{2(\beta-\alpha)}\left(\alpha'' + \beta'^2\right)$$

Following Hehl's approach [2, 3] by redefining pressure and density as

(3.18) $$\bar{p} = p - 2\pi K^2, \quad \bar{\rho} = \rho - 2\pi K^2$$

The field equations reduce to

(3.19) $$8\pi\bar{\rho} = e^{2(\beta-\alpha)}\left(2\beta'' - \alpha'' + \frac{2\beta'}{r} - \beta'^2\right)$$

(3.20) $$8\pi\overline{p_r} = -8\pi\overline{p_z} = e^{2(\beta-\alpha)}\left(\frac{\alpha'}{r} - \beta'^2\right)$$

(3.21) $$8\pi\overline{p_\phi} = e^{2(\beta-\alpha)}\left(\alpha'' + \beta'^2\right)$$

Also the continuity equation becomes

(3.22) $$\frac{d\overline{p_r}}{dr} + \left(\bar{\rho} + \overline{p_r}\right)\beta' - \left(\overline{p_r} - \overline{p_\theta}\right)\left(\beta' - \frac{1}{r}\right)$$

$$-2\overline{p_r}\left(\beta' - \alpha'\right) = 0$$

4. SOLUTION OF THE FIELD EQUATIONS

We have only three independent equations to determine five unknown. Thus the system is indeterminate. For complete determinacy of the system, we require two more conditions

Case 1

Here we assume an equation of state of the form

(4.1) $$\bar{\rho} = a\bar{p}$$

Where 'a' is a constant. This gives an additional equation

(4.2) $$2\beta'' + \frac{2\beta'}{r} - (1-a)\beta'^2 = \frac{a\alpha'}{r} + \alpha''$$

Since our set of equation is still incomplete, we will take a suitable choice of one of the metric coefficients. For this we assume, β in

(4.3) $$\beta = b_1 r^4 + b_2$$

Where b_1 and b_2 are constants. With this value of β, equation (4.2) becomes

(4.4) $$\frac{d^2\alpha}{dr^2}+\frac{ad\alpha}{r\,dr}=30b_1r^2-(1-a)16b_1^2r^6$$

Let $\frac{d\alpha}{dr}=p$, we have

(4.5) $$\frac{dp}{dr}+\frac{a}{r}p=32b_1r^2-16(1-a)b_1^2r^6$$

Equation (4.5) is a linear differential equation in ρ and r Its solution is given by

(4.6) $$p=\frac{d\alpha}{dr}=c_1r^{-a}+\frac{32b_1r^3}{a+3}-\frac{16(1-a)b_1^2r^7}{a+7}$$

Where c_1 is constant of integration. Integration of (4.6) gives

(4.7) $$\alpha=\frac{c_1r^{1-a}}{1-a}+\frac{8b_1r^4}{(3+a)}-\frac{2(1-a)b_1{}^2r8}{(a+7)}+c_2$$

Where c_2 is another constant of integration.

Now we have four arbitrary constant b_1, b_2, c_1 and c_2 which are to be determined by the boundary conditions. If we take $r = r_0$ to be radius of cylinder, we have for $r > r_0$ (i.e. for outside the cylinder) the equation $R_{ij} = 0$ (empty space). A well known solution for Einstein equations for empty space with cylindrical symmetry is that given by Levi-Civita [10] which is given as

(4.8) $$ds^2=-Ar^{-2c(1-c)}(dr^2+dz^2)-r^{2(1-c)}d\phi^2+r^{2c}dt^2$$

Where c and A are constants. We use Licknerowicz boundary conditions, namely that metric potentials are continuous across the surface $r = r_0$. Thus the continuity of $\alpha,\ \alpha'$ and $\beta,\ \beta'$ gives

(4.9) $$b_1=\frac{c}{4r_0{}^4}$$

(4.10) $$b_2=c\log r_0-\frac{c}{4}$$

(4.11) $$c_1=\frac{r_0^{a-1}c}{4(1-a)}\left[\frac{4(1-a)c}{(a+7)r_0}-\frac{2(2a+9)}{a+3}\right]$$

(4.12) $$c_2=\frac{1}{2}\log A+c^2\log r_0-\frac{c^2}{(a+7)}r_0+\frac{c(2a+9)}{2(1-a)(a+3)}$$

$$+\frac{15c}{8(a+3)}+\frac{(1-a)^2}{7(a+7)r_0}c^2$$

Thus we have for the interior of the cylinder, the solution

(4.13) $$\alpha = c\left[\frac{c}{(a+7)r_0} - \frac{(2a+9)}{2(1-a)(a+3)}\right]\in^{1-a}$$

$$+\frac{15c}{8(a+3)}(1+\in^4) - \frac{c^2(1-a)}{8(a+7)r_0}(\in^8 -1) + \frac{1}{2}\log A$$

$$+c^2\log r_0 - \frac{c^2}{(a+7)r_0} + \frac{c(2a+9)}{2(1-a)(a+3)}$$

(4.14) $$\beta = \frac{c}{4}(\in^4 -1) + c\log r_0 \qquad \text{where } \in = \frac{r}{r_0}$$

Also pressure and density are found to be

(4.15) $$8\pi\rho = 16\pi^2H^2e^{-2\alpha} + G\left\{\begin{matrix} 6\in^2 -a(a-1)\left[\frac{c}{a+7} - \frac{(2a+9)r_0}{2(1-a)(a+3)}\right]\in^{-(a+1)} \\ -\frac{45\in^2}{2(a+3)} + \frac{6c(1-a)\in^5}{(a+7)r_0} + 2\in^2 -c\in^6 \end{matrix}\right\}$$

(4.16) $$8\pi p_r = -8\pi p_2 = 16\pi^2H^2e^{-2\alpha} + G\left\{\begin{matrix} (1-a)\left\{\frac{c}{(a+7)r_0} - \frac{(2a+9)}{2(1-a)(a+3)}\right\}\in^{-(a+1)} \\ +\frac{15\in^2}{2(a+3)} - \frac{c(1-a)\in^5}{(a+7)r_0} - c\in^6 \end{matrix}\right\}$$

(4.17) $$8\pi p_\phi = 16\pi^2H^2e^{-2\alpha} + G\left\{\begin{matrix} a(a-1)\left\{\frac{c}{(a+7)r_0} - \frac{(2a+9)}{2(1-a)(a+3)}\right\}\in^{-(a+1)} \\ -\frac{45\in^2}{2(a+3)} + \frac{6c(1-a)\in^5}{(a+7)r_0} + c\in^6 \end{matrix}\right\}$$

Where $$G = \frac{ce^2(\beta-\alpha)}{r_0^{\,2}}$$

CASE II

Here we choose

(4.18) $$\bar{\rho} = a\overline{p_z}$$

Then we get an additional equation

(4.19) $$2\beta'' + \frac{2\beta'}{r} - (1+a)\beta'^2 = \alpha'' - \frac{a\alpha'}{r}$$

Also we assume β same in case I i.e.

(4.20) $$v = b_1' r^4 + b_2'$$

Using (4.20) equation (4.19) becomes

(4.21) $$\frac{d^2\alpha}{dr^2}-\frac{a}{r}\frac{d\alpha}{dr}=32b_1'r^2-16(1+a)b_1'^2r^6$$

Letting $\frac{d\alpha}{dr}=p$, we get

(4.22) $$\frac{dp}{dr}-\frac{a}{r}p=32b_1'r^2-16(1+a)b_1'^2r^6$$

Which is linear differential equation in p and r. It's solution is

(4.23) $$p=c_1'r^a+\frac{32b_1'r^3}{(3-a)}-\frac{16b_1'^2(1+a)r^7}{(7-a)}$$

Where c_1 is constant of integration of (4.4.23) gives

(4.24) $$\alpha=c_1'\frac{r^{a+1}}{a+1}+\frac{8b_1'r^4}{(3-a)}-\frac{2b_1'^2(1+a)r^8}{7-a}$$

Where c_2' is another constant of integration. As in case I using boundary conditions we can find the constants b_1', b_2', c_1', c_2' and also pressure and density can be written similarly.

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