A MAP/PH/1 Production Inventory Model with Different Production Rates

Salini S. Nair,[1] K. P. Jose[2]*

PG & Research Dept. of Mathematics,

St. Peter's College, Kolenchery-682311, Kerala, India.

[1]saliniajithk@gmail.com,[2]kpjspc@gmail.com

Abstract

This paper analyses a production inventory model with retrial of customers under (s, S) policy. The time between additions of two successive items by production to the inventory is exponentially distributed. When the inventory level decreases to s, production starts and the rate of production is higher until the inventory level crosses $s + 1$. Arrival of customers follows a Markovian Arrival Process (MAP) and service times follow a Phase-type (PH) distribution. An arriving customer who identifies the server busy or inventory level zero, proceeds to an orbit of infinite capacity and retry from there. Inter-retrial times follow an exponential distribution. Some important system performance measures of the model are defined and analysed numerically.

Keywords: Different Production Rates, Markovian Arrival Process, Phase-type Distribution, Production Inventory, Retrial.

AMS subject classification: 60K25, 90B05, 91B70

1. Introduction

In recent years, manufactures produce inventories in accordance with the actual demand instead of producing them in anticipation of demand. Production inventory under (s, S) policy can be used to model these type of systems efficiently. Krishnamoorthy and Jose [7] compared three production inventory systems under (s, S) policy with positive service time and retrial of customers by assuming all the underlying distributions to be exponential. They obtained that the model with buffer size equal to the inventoried items is the best profitable model for practical purposes. Baek and Moon[1] studied an (s, S) production inventory system with an attached Markovian service queue. They derived an explicit stationary joint probability in product form. Jose and Salini [4] studied two production inventory systems with positive service time and retrial of customers. They assumed different rates of production depending on the inventory level and analyzed the model numerically.

The Markovian arrival process introduced by M.F. Neuts [11], was indeed a natural generalization of the Poisson Process. MAP take into account the correlation aspect, which arises naturally in many applications in queueing, reliability and inventory models. Kim et al. [6] studied a multi-server retrial queueing system. The arrival of customers followed the Batch Markovian Arrival Process (BMAP) and service times followed a Phase-type distribution. They derived an algorithm to compute the stationary distribution of the system for larger number of servers and illustrated its advantages numerically. Chakravarthy and Neuts [2] considered a multi-server queueing model in which two types of arrivals occurred. The arrival of regular customers was according to a Markovian arrival process and special customers arrived according to a phase type renewal process. They analysed the model numerically in the steady state.

Krishnamoorthy and Viswanath [8] analyzed a production inventory system with positive service time and vacation to the server. The customers arrived according to Markovian arrival process and service times followed a Phase type distribution. They investigated the stability of the system, system state distribution and several performance measures of the system. Karthick et al. [5] analyzed an (s, S) inventory system with two types of customers and Markovian arrival process. They derived various system performance measures in the steady state and total expected cost rate.

The paper is organized as follows: Section 2 presents the mathematical modeling and analysis of the system. Section 3 addresses the stability conditions of the model. Section 4 defines key performance measures. Numerical results and their interpretations are provided in Section 5. Finally, Section 6 concludes the paper with a summary of findings and directions for future research.

The notations used in this article are

S: Maximum inventory level

s: Inventory level at which production starts

$N(t)$: Number of customers in the orbit at time t.

$$C(t) = \begin{cases} 1 \text{ if the server is busy} \\ 0 \text{ if the server is idle} \end{cases}$$

$$F(t) = \begin{cases} 0, \text{ if the production is in OFF mode} \\ 1, \text{ if the production is in ON mode} \end{cases}$$

$J_1(t)$: Phase of the arrival process at time t.

$J_2(t)$: Phase of the service process at time t.

e : a column vector of 1's of appropriate order.

2. Mathematical Modeling and Analysis

We consider a production inventory system with retrials under (s,S) policy. When the inventory level reduces to s, production starts and stops when the inventory level reaches back to S. The arrival of customers is according to a Markovian arrival process with representation (D_0,D_1). The service times follow Phase type distribution with representation (η,T). The time between additions of two items to the inventory is exponentially distributed with parameter β. When the inventory level reduces to s, production starts and the rate of production is $\alpha\beta, \alpha \in [1,k]$ where k is a finite value greater than 1, until the inventory level crosses $s+1$. An arriving customer, who notices the inventory level zero or server busy, proceeds to an orbit of infinite capacity with probability γ. An orbiting customer may retry from there and inter retrial times are exponentially distributed with parameter $i\theta$ when there are i customers in the orbit. If a retrial customer, who finds the inventory level zero or server busy, returns back to the orbit with probability δ.

Now $\{X(t), t \geq 0$, where $X(t) = (N(t), C(t), F(t), I(t), J_1(t), J_2(t))$, is a level dependent quasi birth death process on the state space

$\{(i,0,0,j,k); i \geq 0, s+1 \leq j \leq S, 1 \leq k \leq m_1\} \cup \{(i,0,1,j,k);\ i \geq 0, 0 \leq j \leq S-1, 1 \leq k \leq m_1\} \cup \{(i,1,0,j,k,l); i \geq 0, s+1 \leq j \leq S, 1 \leq k \leq m_1, 1 \leq l \leq m_2\} \cup (i,1,1,j,k,l);\ i \geq 0, 1 \leq j \leq S-1, 1 \leq k \leq m_1, 1 \leq l \leq m_2\}$

The Markov chain governing the system dynamics is characterized by the following infinitesimal generator matrix.

$$Q = \begin{bmatrix} A_{1,0} & A_0 & & & \\ A_{2,1} & A_{1,1} & A_0 & & \\ & A_{2,2} & A_{1,2} & A_0 & \\ & & A_{2,3} & A_{1,3} & A_0 \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

where the blocks A_0, $A_{1,i}$ $(i \geq 0)$ and $A_{2,i}$ $(i \geq 1)$ are square matrices of order $[(2S-s)m_1 + (2S-s-1)m_1m_2]$ and they are given by

$$A_0 = \begin{array}{c} \underline{0,0} \\ \underline{0,1} \\ \underline{1,0} \\ \underline{1,1} \end{array} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & C1 & 0 & 0 \\ 0 & 0 & C2 & 0 \\ 0 & 0 & 0 & C3 \end{bmatrix}$$

3

$$A_{1,i} = \begin{matrix} \underline{0,0} \\ \underline{0,1} \\ \underline{1,0} \\ \underline{1,1} \end{matrix} \begin{bmatrix} B1 & 0 & B2 & 0 \\ B3 & B4 & 0 & B5 \\ B6 & B7 & B8 & 0 \\ 0 & B9 & B10 & B11 \end{bmatrix}$$

$$A_{2,i} = \begin{matrix} \underline{0,0} \\ \underline{0,1} \\ \underline{1,0} \\ \underline{1,1} \end{matrix} \begin{bmatrix} 0 & 0 & G1 & 0 \\ 0 & G2 & 0 & G3 \\ 0 & 0 & G4 & 0 \\ 0 & 0 & 0 & G5 \end{bmatrix}$$

$(p,q)^{th}$ element of the matrices contained in A_0, $A_{1,i}$ and $A_{2,i}$ are given by

$$[C_1]_{pq} = \begin{cases} \gamma D_1, & p = q = 1 \\ 0, & otherwise \end{cases}$$

$$[C_2]_{pq} = \begin{cases} \gamma D_1 \otimes I_{m_2}, & 1 \le p \le (S-s), q = p \\ 0, & otherwise \end{cases}$$

$$[C_3]_{pq} = \begin{cases} \gamma D_1 \otimes I_{m_2}, & 1 \le p \le (S-1), q = p \\ 0, & otherwise \end{cases}$$

$$[B_1]_{pq} = \begin{cases} \psi, & 1 \le p \le S - S, \ q = p \\ 0, & otherwise \end{cases}$$

$$[B_2]_{pq} = \begin{cases} D_1 \otimes \eta, & 1 \le p \le (S-s), \ q = p \\ 0, & otherwise \end{cases}$$

$$[B_3]_{pq} = \begin{cases} \beta I_{m_1}, & p = q = S \\ 0, & otherwise \end{cases}$$

$$[B_4]_{pq} = \begin{cases} \Gamma_0, p = q = 1 \\ \Gamma_1, \ 2 \le p \le (s+1), \ q = p \\ \Gamma_2, (s+2) \le p \le S, \ q = p \\ \alpha\beta I_{m_1,} 1 \le p \le s+1, \ q = p+1 \\ \beta I_{m_1} (s+2) \le p \le S-1, \ q = p+1 \\ 0, \ otherwise \end{cases}$$

$$[B_5]_{pq} = \begin{cases} D_1 \otimes \eta, & 2 \le p \le S), \ q = p \\ 0, & otherwise \end{cases}$$

$$[B_6]_{pq} = \begin{cases} I_{m_1} \otimes T^0, & 2 \le p \le (S-s), \ q = p-1 \\ 0, & otherwise \end{cases}$$

$$[B_7]_{pq} = \begin{cases} I_{m_1} \otimes T^0, & p = 1, \ \ q = s+1 \\ 0, & otherwise \end{cases}$$

$$[B_8]_{pq} = \begin{cases} \Delta, & 1 \le p \le (S-s), \ \ q = p \\ 0, & otherwise \end{cases}$$

$$[B_9]_{pq} = \begin{cases} I_{m_1} \otimes T^0, & 1 \le p \le (S-1), \ \ q = p \\ 0, & otherwise \end{cases}$$

$$[B_{10}]_{pq} = \begin{cases} \beta I_{m_1 m_2}, & p = (S-1), \ \ q = S-s \\ 0, & otherwise \end{cases}$$

$$[B_{11}]_{pq} = \begin{cases} \Omega_1, 1 \le p \le s, & q = p \\ \Omega_2, (s+1) \le p \le S-1, & q = p \\ \alpha\beta I_{m_1 m_2}, 1 \le p \le s, & q = p+1 \\ \beta I_{m_1 m_2}, (s+1) \le p \le S-2, & q = p+1 \\ 0, & otherwise \end{cases}$$

$$[G_1]_{pq} = \begin{cases} i\theta I_{m_1} \otimes \eta, & 1 \le p \le (S-s), \ \ q = p \\ 0, & otherwise \end{cases}$$

$$[G_2]_{pq} = \begin{cases} i\theta(1-\delta) I_{m_1}, & p = q = 1 \\ 0, & otherwise \end{cases}$$

$$[G_3]_{pq} = \begin{cases} i\theta I_{m_1} \otimes \eta, & 2 \le p \le S, \ \ q = p-1 \\ 0, & otherwise \end{cases}$$

$$[G_4]_{pq} = \begin{cases} i\theta(1-\delta) I_{m_1 m_2}, & 1 \le p \le (S-s), \ \ q = p \\ 0, & otherwise \end{cases}$$

$$[G_5]_{pq} = \begin{cases} i\theta(1-\delta) I_{m_1 m_2}, & 1 \le p \le (S-1), \ \ q = p \\ 0, & otherwise \end{cases}$$

where $\psi = -(D_1 \otimes \eta)$- $i\theta I_{m_1} \otimes \eta$

$\Gamma_0 = -\gamma D_1$-$i\theta(1-\delta) I_{m_1} - \alpha\beta I_{m_1}$

$\Gamma_1 = -(D_1 \otimes \eta)$- $i\theta(I_{m_1} \otimes \eta) - \alpha\beta I_{m_1}$

$\Gamma_2 = -(D_1 \otimes \eta)$- $i\theta(I_{m_1} \otimes \eta) - \beta I_{m_1}$

$\Delta = -I_{m_1} \otimes T^0 - i\theta(1-\delta) I_{m_1 m_2} - \gamma D_1 \otimes I_{m_2}$

$\Omega_1 = -I_{m_1} \otimes T^0 - \alpha\beta I_{m_1 m_2} - i\theta(1-\delta) I_{m_1 m_2} - \gamma D_1 \otimes I_{m_2}$

$\Omega_2 = -I_{m_1} \otimes T^0 - \beta I_{m_1 m_2} - i\theta(1-\delta) I_{m_1 m_2} - \gamma D_1 \otimes I_{m_2}$

$A_{1,i}, i \ge 0$ governs transitions from i to i; A_0, transitions from i to $i+1$; $A_{2,i}, i \ge 1$, transitions from i to $i-1$. To ensure the tractability of the infinite-dimensional generator matrix, Neuts-Rao [10] truncation method is employed. Under this approach, the matrices $A_{1,i}$ and $A_{2,i}$ are assumed to be constant for sufficiently large i.

5

3. System Stability

To analyze the stability of the system, we employ a Lyapunov test function as proposed by Falin and Templeton [3], defined by $\varphi(r) = i$, if r is a state in the level i. The mean drift y_r for any r belonging to the level $i \geq 1$ is given by

$$y_r = \sum_{p \neq r} q_{rp}(\varphi(p) - \varphi(r))$$
$$= \sum_u q_{ru}\left(\varphi(u) - \varphi(r)\right) + \sum_v q_{rv}\left(\varphi(v) - \varphi(r)\right) + \sum_w q_{rw}\left(\varphi(w) - \varphi(r)\right)$$

where u, v, w vary over the states belonging to the levels $(i-1)$, i and $(i+1)$ respectively. Then by the definition of φ, $\varphi(u) = i-1$, $\varphi(v) = i$ and $\varphi(w) = i+1$ so that

$$y_r = -\sum_u q_{ru} + \sum_w q_{rw}$$
$$= \begin{cases} -i\theta(1-\delta) + \gamma(D_1 e)_k, \ r = (i,0,1,0,k), 1 \leq k \leq m_1 \\ -i\theta, \ r = (i,0,0,j,k), (s+1) \leq j \leq S, 1 \leq k \leq m_1 \\ -i\theta, \ r = (i,0,1,j,k), 1 \leq j \leq S-1, 1 \leq k \leq m_1 \\ -i\theta(1-\delta) + (\gamma(D_1 e) \otimes e_{m_2})_{(k-1)m_1+l}, \ r = (i,1,0,j,k,l), \\ (s+1) \leq j \leq S, 1 \leq k \leq m_1, 1 \leq l \leq m_2 \\ -i\theta(1-\delta) + (\gamma(D_1 e) \otimes e_{m_2})_{(k-1)m_1+l}, \ r = (i,1,1,j,k,l) \\ 1 \leq j \leq S-1, 1 \leq k \leq m_1, 1 \leq l \leq m_2 \end{cases}$$

Since $(1 - \delta) > 0$, for any $\varepsilon > 0$, it is possible to choose a sufficiently large level N' such that the mean drift $y_r < -\varepsilon$ for any r belonging to the level $i \geq N'$. Therefore, by the stability criterion established by Tweedie [12], the system under consideration is stable.

3.1 Rate Matrix R and Truncation Level N

In order to find R, we use iterative method. Denote the sequence of R by $\{R_n(N)\}$ and is defined by $R_0(N) = 0$ and $R_{n+1}(N) = (-R_n^2(N)A_2(N) - A_0(N))A_1^{-1}(N))$. The value of N must be chosen such that $|\eta(N) - \eta(N+1)| < \varepsilon$, where ε is an arbitrarily small value and $\eta(R)$, spectral radius of $R(N)$. For detailed discussion of selection of value of N, one can refer to Neuts [9].

4. System Performance Measures

Let $x = (x_0, x_1, \dots, x_{N-1}, x_N, \dots)$ denotes the steady state probability vector where x_i represents the long run probability that the system is in state i.

a) Overall rate of retrials (ORR) is given by,

$$ORR = \theta\left(\sum_{i=1}^{\infty} i x_i\right) e$$

b) Successful rate of retrials (SRR) is given by,

$$SRR = \theta \sum_{i=1}^{\infty} \left(\sum_{j=s+1}^{S} i y_{i,0,0,j} e_{m_1} + \sum_{j=1}^{S-1} i y_{i,0,1,j}\, e_{m_1} \right)$$

c) Server busy probability, SBP, is given by,

$$SBP = \sum_{i=0}^{\infty} \sum_{j=s+1}^{S} y_{i,1,0,j}\, e_{m_1 m_2} + \sum_{i=0}^{\infty} \sum_{j=1}^{S-1} y_{i,1,1,j}\, e_{m_1 m_2}$$

d) Expected number of crossovers in one cycle, ECC, is given by,

$$ECC = \sum_{i=0}^{\infty} \alpha\beta y_{i,1,1,s-1} + \sum_{i=0}^{\infty} y_{i,1,1,s+1}\, (e_{m_1} \otimes T^0)$$

5. Numerical Results and Interpretations

We analyze the model by considering the performance measures such as overall and successful rate of retrials (ORR and SRR), server busy probability (SBP) and expected number of crossovers in one cycle (ECC). The values of ORR, SRR, SBP and ECC by varying the parameters $\alpha, \beta, \gamma, \delta$ and θ are given in the following tables.

Consider the following parameter values.

$m_1 = 2, m_2 = 2, D_0 = \begin{bmatrix} -2.1 & 1.0 \\ 1.08 & -3.1 \end{bmatrix}, D_1 = \begin{bmatrix} 0.1 & 1.0 \\ 1.08 & 1.1 \end{bmatrix}$

$\eta = (0.5, 0.5), T = \begin{bmatrix} -6 & 3 \\ 1 & -4 \end{bmatrix}, T_0 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$

In Table1, the overall retrial rate (ORR) shows a decreasing trend, while the successful retrial rate (SRR) increases. This behavior is attributed to the rise in production rate, which results in fewer customers entering the orbit, thereby reducing ORR. At the same time, a higher production rate enhances the likelihood of successful retrials, leading to an increase in SRR. Additionally, this improvement in production boosts the system busy probability (SBP) and reduces the expected cost overs in one cycle (ECC), as observed in Table 1.

In contrast, Table2 and Table3 show an upward trend in all performance measures. This is due to the increased probabilities of both primary and retrial customers entering the orbit, which in turn raises the number of orbiting customers. As a result, ORR, SRR, SBP and ECC all exhibit an increase. Table 4 indicates that with an increase in the retrial rate, all performance indicators rise except for ECC, which declines.

α	ORR	SRR	SBP	ECC
1.1	7.7102	0.9143	0.5252	0.1710
1.2	7.6013	0.9171	0.5297	0.1596
1.3	7.5230	0.9205	0.5333	0.1505
1.4	7.4655	0.9240	0.5361	0.1430
1.5	7.4222	0.9273	0.5385	0.1368
1.6	7.3890	0.9305	0.5405	0.1316
1.7	7.3631	0.9333	0.5421	0.1270
1.8	7.3426	0.9359	0.5435	0.1229
1.9	7.3261	0.9382	0.5447	0.1192

$S = 15; s = 5;\ \gamma = 0.6; N = 50; \theta = 1.5; \delta = 0.8; \beta = 1.5.$

Table 1: Variations in α

γ	ORR	SRR	SBP	ECC
0.1	1.1022	0.1543	0.3748	0.0326
0.2	2.2784	0.3111	0.4077	0.0538
0.3	3.5172	0.4685	0.4408	0.0758
0.4	4.8050	0.6244	0.4735	0.0983
0.5	6.1265	0.7769	0.5054	0.1208
0.6	7.4655	0.9240	0.5361	0.1430
0.7	8.8057	1.0640	0.5654	0.1645
0.8	10.1319	1.1956	0.5929	0.1849
0.9	11.4309	1.3179	0.6185	0.2040

$S = 15; s = 5;\ \alpha = 1.4; N = 50; \theta = 1.5; \delta = 0.8; \beta = 1.5$

Table 2: Variations in γ

δ	ORR	SRR	SBP	ECC
0.1	2.4020	0.4111	0.4300	0.0520
0.2	2.6544	0.4444	0.4371	0.0559
0.3	2.9674	0.4841	0.4455	0.0608
0.4	3.3664	0.5324	0.4557	0.0672
0.5	3.8929	0.5926	0.4683	0.0759
0.6	4.6212	0.6702	0.4845	0.0884
0.7	5.6983	0.7746	0.5060	0.1081
0.8	7.4655	0.9240	0.5361	0.1430
0.9	10.974	1.1549	0.5797	0.2211

$S = 15; s = 5;\ \alpha = 1.4; N = 50; \theta = 1.5; \gamma = 0.6; \beta = 1.5$

Table 3: Variations in δ

θ	ORR	SRR	SBP	ECC
1.1	9.9783	1.4653	0.6492	0.2260
1.2	10.6861	1.4963	0.6562	0.2236
1.3	11.3836	1.5253	0.6628	0.2211
1.4	12.0716	1.5525	0.6689	0.2186
1.5	12.7507	1.5781	0.6747	0.2161
1.6	13.4216	1.6023	0.6802	0.2137
1.7	14.0846	1.6253	0.6855	0.2113
1.8	14.7402	1.6472	0.6904	0.2089
1.9	15.3889	1.6680	0.6952	0.2066

$S = 15; s = 5;\ \alpha = 1.4; N = 50; \delta = 0.8; \gamma = 0.6; \beta = 1.5$

Table 4: Variations in θ

8

Concluding Remarks

In this paper, we considered a production–inventory system characterized by multiple production rates and a retrial mechanism for customers who did not receive immediate service. Our analysis focused on establishing the stability conditions of the underlying stochastic model and examining how system parameters influenced its long-run performance. Several key steady-state performance measures were derived analytically and evaluated through numerical experiments. The proposed framework could be further generalized by incorporating a Batch Markovian Arrival Process (BMAP), allowing the model to capture arrival patterns found in real production and service systems.

References

[1] Baek, J. W., & Moon, S. K. (2016). A production-inventory system with a Markovian service queue and lost sales. *Journal of the Korean Statistical Society*, *45*(1), 14-24.

[2] Chakravarthy, S. R., & Neuts, M. F. (2014). Analysis of a multi-server queueing model with MAP arrivals of regular customers and phase type arrivals of special customers. *Simulation Modelling Practice and Theory*, *43*, 79-95.

[3] Falin, G. I. & Templeton, J. G., & (1997). *Retrial queues*. Vol 75 CRC Press

[4] Jose, K. P., & Nair, S. S. (2017). Analysis of two production inventory systems with buffer, retrials and different production rates. *Journal of Industrial Engineering International*, *13*(3), 369-380.

[5] Karthick, T., Sivakumar, B., & Arivarignan, G. (2015). An inventory system with two types of customers and retrial demands. *International Journal of Systems Science: Operations & Logistics*, *2*(2), 90-112.

[6] Kim, C. S., Mushko, V. V., & Dudin, A. N. (2012). Computation of the steady state distribution for multi-server retrial queues with phase type service process. *Annals of Operations research*, *201*(1), 307-323.

[7] Krishnamoorthy, A., & Jose, K. P. (2008). Three production inventory systems with service, loss and retrial of customers. *International journal of information and management sciences*, *19*(3), 367-389.

[8] Krishnamoorthy, A., & Narayanan, V. C. (2011). Production inventory with service time and vacation to the server. *IMA Journal of Management Mathematics*, *22*(1), 33-45.

[9] Neuts, M. F. (1994). *Matrix-geometric solutions in stochastic models: an algorithmic approach*. Courier Corporation.

[10] Neuts, M. F., & Rao, B. M. (1990). Numerical investigation of a multiserver retrial model. *Queueing systems*, *7*(2), 169-189.

[11] Neuts, M. F. (1979). A versatile Markovian point process. *Journal of Applied Probability*, *16*(4), 764-779.

[12] Tweedie, R. L. (1975, July). Sufficient conditions for regularity, recurrence and ergodicity of Markov processes. In *Mathematical Proceedings of the Cambridge Philosophical Society* (Vol. 78, No. 1, pp. 125-136). Cambridge University Press.

[13] Yadavalli, V. S., Sivakumar, B., Arivarignan, G., & Adetunji, O. (2011). A multi-server perishable inventory system with negative customer. *Computers & Industrial Engineering*, *61*(2), 254-273.