Brief Study of Special Type of Set Named Midpoint Convex Set

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Abstract : In this article, a midpoint convex set has been defined which is generalisation of convex set in this sense that every convex set is a midpoint convex set but a midpoint convex set may not be necessarily convex set. Some interesting Theorems also stablished, and proved.

Translate of a set also defined in this paper.

Keywords : Convex, Midpoint convex, Linear space.

1. Introduction :

Midpoint Convex Set :

The line segment joining two points x and y of a linear space in the set of all points of the form $ax + by$ with a and b are non-negative real numbers such that $a + b = 1$, or equivalently, the set of all points $ax + (1 - a)y$ with a real and $0 \leq a \leq 1$.

This set is denoted by $[x : y]$. A set in a linear space L is convex if, whenever it contains x and y, it also contains $[x : y]$. [1]

A subset A of a linear space L is midpoint convex if and only if $\frac{1}{2}(x+y)$ is in A whenever x and y are in A.[2]

Thus it is clear that a convex set is always a midpoint convex but a midpoint convex set may not be necessarily convex set.

Theorem (2.I) :

Let A be a midpoint convex set of a linear space L. Then $\frac{1}{2}A+\frac{1}{2}A = A$.

Proof : Let a be an element of A. Then $a = \frac{1}{2}a+\frac{1}{2}a \in \frac{1}{2}A+\frac{1}{2}A$.

Thus $a \in A \Rightarrow a \in \frac{1}{2}A+\frac{1}{2}A$

Hence $A \subseteq \frac{1}{2}A+\frac{1}{2}A$ … (2.1)

Conversely, let x be an element of $\frac{1}{2}A+\frac{1}{2}A$.

Then we can write $x = \frac{1}{2}a_1+\frac{1}{2}a_2$ where $a_1, a_2 \in A$.

But since A is a midpoint convex,

$\therefore$ $\frac{1}{2}a_1+\frac{1}{2}a_2 = \frac{1}{2}(a_1+a_2) \in A$.

Thus x is in A.

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Hence $x \in \frac{1}{2}A + \frac{1}{2}A \Rightarrow x \in A$

Therefore

$$\frac{1}{2}A + \frac{1}{2}A \subseteq A \qquad \ldots (2.2)$$

From (2.1) and (2.2), we have

$A = \frac{1}{2}A + \frac{1}{2}A.$

Theorem (2.II) :

If A_1, A_2 are midpoint convex and if λ_1, λ_2 are scalars then $\lambda_1 A_1 + \lambda_2 A_2$ is also midpoint convex.

Proof : Let x_1, x_2 be elements of $\lambda_1 A_1 + \lambda_2 A_2$.

Then we can write

$x_1 = \lambda_1 A_1 + \lambda_2 A_2$, where $a_1 \in A_1$ and $a_2 \in A_2$

and $x_2 = \lambda_1 a_1' + \lambda_2 a_2'$, where $a_1' \in A_1$ and $a_2' \in A_2$

But since A_1 and A_2 are midpoint convex sets,

so $\frac{1}{2}(a_1 + a_1')$ in an element of A_1

and $\frac{1}{2}(a_2 + a_2')$ in an element of A_2

Now $\frac{1}{2}(x_1 + x_2) = \frac{1}{2}(\lambda_1 a_1 + \lambda_2 a_2 + \lambda_1 a_1' + \lambda_2 a_2')$

$= \lambda_1 \cdot \frac{1}{2}(a_1 + a_1') + \lambda_2 \cdot \frac{1}{2}(a_2 + a_2')$

$\in \lambda_1 A_1 + \lambda_2 A_2$

Thus $\lambda_1 A_1 + \lambda_2 A_2$ is a midpoint convex set.

Theorem (2.III) :

The intersection of a family of midpoint convex sets is *a* midpoint convex.

Proof : Let $\{Ai\}$ be a family of midpoint convex sets in *a* linear space *L*. Then we will show that $\bigcap_i Ai$ is also a midpoint convex

Let x, y be elements of $\bigcap_i Ai$.

Then for each i, $x \in Ai$ and $y \in Ai$

But since Ai is a midpoint contex for each i,

Therefore $\frac{1}{2}(x + y)$ also belongs to Ai for each i

Hence $\frac{1}{2}(x + y) \in \bigcap_i Ai$

Therefore $\bigcap_i Ai$ is a midpoint convex set.

Theorem (2.IV) :

Let $\{Ai\}$ be a family of midpoint convex sets such that $A_1 \subseteq A_2 \subseteq A_3 \subseteq ....$, that is $Ai \subseteq Aj$ whenever $i \le j$. Then $\bigcup_i Ai$ is also midpoint convex.

Proof : Let *x*, *y* be the elements of $\bigcup_i Ai$. Then there exist *i*, *j* such that *x* belongs to Ai and *y* belongs to Aj . Let us assume that $i \le j$.

Now since $i \le j$, $Ai \subseteq Aj$

Hence $x \in Ai \Rightarrow x \in Aj$

Therefore $x, y \in Aj$.

Since *Aj* is midpoint convex, $\frac{1}{2}(x+y) \in Aj$

Therefore $\frac{1}{2}(x+y) \in \bigcup_i Ai$

Thus $x, y \in \bigcup_i Ai \Rightarrow \frac{1}{2}(x+y) \in \bigcup_i Ai$

Therefore $\bigcup_i Ai$ is a midpoint convex set.

Theorem (2.V) :

Let *T* be a linear transformation from a linear space *X* to a linear space *Y*. Then the image of each midpoint convex set is *X* is a midpoint convex set in *Y* and The inverse image of each midpoint convex set in *Y* is a midpoint convex set in *X*.

Proof : Let *A* be a midpoint convex set in *X*, Then $T(A) \subseteq Y$. Let z_1, z_2 be the elements of *T*(*A*). Then there exists x_1 and x_2 in *A* such that $T(x_1) = z_1$ and $T(x_2) = z_2$

Since *A* is a midpoint convex, Therefore $\frac{1}{2}(x_1 + x_2)$ is an element of *A*.

$$\text{Now } \frac{1}{2}(z_1+z_2) = \frac{1}{2}(Tx_1 + Tx_2) = \frac{1}{2}T(x_1+x_2)$$

$$= T\left(\frac{1}{2}(x_1+x_2)\right) \in T(A).$$

Hence *T*(*A*) is midpoint convex.

Next, Let *B* be a midpoint convex in *Y*.

Then $T^{-1}(B) = \{x : T(x) \in B\} \subseteq X.$

Let x_1 and x_2 be the elements of $T^{-1}(B)$: Then exist elements z_1 and z_2 in *B* such that $z_1 = T(x_1)$ and $z_2 = T(x_2)$

Since *B* is a midpoint convex

$$\frac{1}{2}(z_1+z_2) \in B \Rightarrow \frac{1}{2}(Tx_1+Tx_2) \in B$$

$$\Rightarrow T\left(\frac{1}{2}(x_1+x_2)\right) \in B$$

$$\Rightarrow \frac{1}{2}(x_1+x_2) \in T^{-1}(B)$$

$\Rightarrow$ $T^{-1}(B)$is midpoint convex.

Defination :

Let L be a linear space and $A \subseteq L$. If x is an element of L, then $x + A$ defined by $x + A = \{x + a : a \in A\}$ is called a translate of A.

Theorem (2.VI) :

Let A be a subeset of a linear space L. If A is a midpoint convex, then any translate of A is also midpoint convex.

Proof : Let $x + A$ be any translate of A, where $x \in L$. Let y_1 and y_2 be elements of $x + A$. Then there exist a_1 and a_2 in A such that

$y_1 = x + a_1,\ y_2 = x + a_2$

Now $\frac{1}{2}(y_1 + y_2) = \frac{1}{2}(x + a_1 + x + a_2) = x + \frac{1}{2}(a_1 + a_2)$

Since A is midpoint convex, $\frac{1}{2}(a_1 + a_2)$ is in A.

Hence $\frac{1}{2}(y_1 + y_2)$ belongs to $x + A$.

Therefore $x + A$ is a midpoint convex set.

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