# On b-coloring of Tadpole graphs

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**Abstract**

The b-chromatic number of G, denoted by $\varphi(G)$, is the maximum k for which G has a b-coloring by k colors. A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex $x_i$ having neighbors in all the other k-1 color classes. Such a vertex $x_i$ is called a b-dominating vertex, and the set of vertices $\{x_1, x_2 \ldots x_k\}$ is called a b-dominating system. In this paper, we investigate the b-chromatic number of Line graph, Middle graph and Total graph of Tadpole graph, denoted by $L(T_{m,n})$, $M(T_{m,n})$ and $T(T_{m,n})$ respectively.

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**Keywords:** Line graph, Middle graph, Total graph, Tadpole graph, b-coloring, b-chromatic number.

## 1. Introduction

Graph theory is the theory of graphs dealing with nodes and connections or vertices and edges. The subject has experienced explosive growth, due in large measure to its role as an essential structure underpining modern applied mathematics. Configurations of nodes and connections has great diversity of applications. They may represent physical networks, such as electrical circuits, roadways, or organic molecules. They are also used in representing less

tangiable interactions as might occur in ecosystem, sociological relationships, database, or in the flow of control in a computer program. It is a fastest growing field in Mathematics mainly because of its applications in distinct areas. There are plenty of works have been done in different topics, like decomposition, domination, factoring, orienting, coloring etc, in the past decades. In Graph theory, a graph $G = (V, E)$, consists of two sets V and E. The elements of V are called vertices or nodes. The elements of E are called edges or connections. In this paper, all the graphs considered as loop less, undirected and having no multiple edges. Graph coloring deals with the general and widely applicable concept of partitioning the underlying set of a structure into parts, each of which satisfies a given requirement. Here coloring means a vertex coloring of a graph. A k-coloring of a graph $G = (V, E)$is a mapping $C : V \to S$ where S is a set of k colors; thus k-coloring is an assignment of k colors to the vertices of G. Usually, the set S of colors is taken to be $\{1, 2, \ldots k\}$. A coloring C is proper if no two adjacent vertices are assigned the same color. Only loop less graphs admits proper coloring. The minimum k for which a graph G is k-colorable is called its chromatic number, and denoted $\chi(G)$. If $\chi(G) = k$, the graph G is said to be k-chromatic. From past few decades, a large number of researchers have been showed their interest to obtain different chromatic numbers for the Line, Middle and Total Graphs. The b-chromatic number of G, denoted by $\varphi(G)$, is the maximum k for which G has a b-coloring by k colors. A b-coloring of G by k colors is a proper k-coloring of the vertices of G such that in each color class i there exists a vertex $x_i$ having neighbors in all the other k-1 color classes. Such a vertex $x_i$ is called a b-dominating vertex, and the set of vertices $\{x_1, x_2 \ldots x_k\}$ is called a b-dominating system. The b-coloring was introduced by R.W.Irving and D.F.Manlove in [7]. They proved that determining $\varphi(G)$ is NP-hard in general and polynomial for trees.
The $(m, n)$-tadpole graph is a special type of graph consisting of a cycle graph on m (*at least* 3) vertices and a path graph on n vertices, connected with a bridge.In this paper we investigate the b-chromatic number of graphs obtained by many different Graph Constructions from a Tadpole graph.

## 2. b- Coloring of Line Graph of Tadpole Graph

**Theorem 2.1.** *For a tadpole graph $T_{m,n}$, $m\,(3 \leq m \leq 5)$, $n \geq 1$, the b-chromatic number of the line graph $L\,(T_{m,n})$is* 3.

$$i.e., \varphi\,[L\,(T_{m,n})] = 3.$$

**Proof :**

For a Tadpole graph $T_{m,n}$, $m\,(3 \leq m \leq 5)$ and $n \geq 1$, let $V\,(T_{m,n}) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and $E\,(T_{m,n}) = \{m_i : 1 \leq i \leq n\} \cup \{k_i : 1 \leq i \leq n\}$.
Let vertices $m_i^{'}$ and $k_i^{'}$, $1 \leq i \leq n$ represents the edges $m_i$ and $k_i$ respectively. By the definition of the line graph corresponding to $T_{m,n}$, assign a proper 3 coloring to $V\,[L\,(T_{m,n})]$ as follows:
**Case 1:** If $m \equiv 0\,(mod\ 3)\,, n \geq 1$
Assign colors $1, 2, 3, 1, 2, 3, ..., 1, 2, 3$ to consecutive vertices $m_i^{'}$, $1 \leq i \leq n$ and colors $2, 3, 2, 3, ..., 2, 3$ to consecutive vertices of $k_i^{'}$, $1 \leq i \leq n$. An easy check shows that it is a proper b-coloring.
**Case 2:** If $m \equiv 1\,(mod\ 3)\,, n \geq 1$
Assign colors $1, 3, 1, 2, 1, 3, 1, 2..., 1, 3, 1, 2$ to consecutive vertices $m_i^{'}$, $1 \leq i \leq n$ and colors $3, 2, 3, 2, ..., 3, 2$ to consecutive vertices of $k_i^{'}$, $1 \leq i \leq n$. An easy check shows that it is a proper b-coloring.
**Case 3:** If $m \equiv 2\,(mod\ 3)\,, n \geq 1$
Assign colors $1, 2, 3, 1, 2, 1, 2, 3, 1, 2, ..., 1, 2, 3, 1, 2$ to consecutive vertices $m_i^{'}$, $1 \leq i \leq n$ and colors $3, 2, 3, 2..., 3, 2$ to consecutive vertices of $k_i^{'}$, $1 \leq i \leq n$. An easy check shows that it satisfies the b-coloring.
From the above cases it follow that $\varphi\,[L\,(T_{m,n})] = 3$.

## 3. b- Coloring of Middle Graph of Tadpole Graph

**Theorem 3.1.** *For a tadpole graph, $T_{m,n}$, $m\,(3 \leq m \leq 5)$, $n \geq 3$, the b-chromatic number of the middle graph $M\,(T_{m,n})$is* 5.

$$i.e., \varphi\,[M\,(T_{m,n})] = 5.$$

**Proof :**

For a Tadpole graph $T_{m,n}$, $m\,(3 \le m \le 5)$ and $n \ge 3$, let $V\,(T_{m,n}) = \{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\}$ and
$E\,(T_{m,n}) = \{m_i : 1 \le i \le n\} \cup \{k_i : 1 \le i \le n\}$
Let vertices $m_i^{'}$ and $k_i^{'}$, $1 \le i \le n$ represents the newly introduced vertices between the edges $m_i$ and $k_i$ respectively.
By the definition of the middle graph
$V\,[M\,(T_{m,n})] = V\,(T_{m,n}) \;\cup\; E\,(T_{m,n}) = \{x_i : 1 \le i \le n\} \cup \{y_i : 1 \le i \le n\} \cup \left\{m_i^{'} : 1 \le i \le n\right\} \cup \left\{k_i^{'} : 1 \le i \le n\right\}$.
Assign a proper 5 coloring to $V\,[M\,(T_{m,n})]$ as follows:
**Case 1:** If $m \equiv 0\,(mod\ 3)$
For $1 \le i \le n$, assign colors $2, 3, 1, 2, 3, 1, ..., 2, 3, 1$ to consecutive vertices of $m_i^{'}$, colors $4, 4, 5, 4, 4, 5, ..., 4, 4, 5$ to consecutive vertices of $x_i$, colors $5, 4, 3, 5, 4, 3, ..., 5, 4, 3$ to consecutive vertices of $k_i^{'}$, colors $1, 2, 1, 2, ..., 1, 2$ to consecutive vertices of $y_i$. This shows that this is a maximum b-coloring with 5 colors.
**Case 2:** If $m \equiv 1\,(mod\ 3)$
For $1 \le i \le n$, assign colors $1, 2, 4, 3, 1, 2, 4, 3, ..., 1, 2, 4, 3, 1$ to consecutive vertices of $m_i^{'}$, colors $4, 3, 5, 1, 4, 3, 5, 1, ..., 4, 3, 5, 1$ to consecutive vertices of $x_i$, colors $5, 3, 4, 5, 3, 4, ..., 5, 3, 4$ to consecutive vertices of $k_i^{'}$, colors $2, 1, 2, 1, ..., 2, 1$ to consecutive vertices of $y_i$. This shows that this is a maximum b-coloring with 5 colors.
**Case 3:** If $m \equiv 2\,(mod\ 3)$
For $1 \le i \le n$, assign colors $1, 2, 3, 4, 5, 1, 2, 3, 4, 5..., 1, 2, 3, 4, 5$ to consecutive vertices of $m_i^{'}$, colors $3, 4, 5, 1, 2, 3, 4, 5, 1, 2..., 3, 4, 5, 1, 2$ to consecutive vertices of $x_i$, colors $2, 3, 2, 3, ..., 2, 3$ to consecutive vertices of $k_i^{'}$, colors $1, 4, 1, 4, ..., 1, 4$ to consecutive vertices of $y_i$. This shows that this is a maximum b-coloring with 5 colors.
From the above cases it follow that $\varphi\,[M\,(T_{m,n})] = 5$.

## 4. b- Coloring of Total Graph of Tadpole Graph

**Theorem 4.1.** *For a tadpole graph,*$T_{m,n}$, $m\,(3 \le m \le 5)$, $n \ge 1$ *the b-chromatic number of the Total graph* $T\,(T_{m,n})$ *is* 5.

$$i.e., \varphi\,[T\,(T_{m,n})] = 5.$$

**Proof :**

For a Tadpole graph $T_{m,n}$, $m\,(3 \leq m \leq 5)$ and $n \geq 3$, let $V\left(T_{m,n}\right) = \{x_i : 1 \leq i \leq n\} \cup \{y_i : 1 \leq i \leq n\}$ and $E\left(T_{m,n}\right) = \{m_i : 1 \leq i \leq n\} \cup \{k_i : 1 \leq i \leq n\}$.
Let vertices $m_i^{'}$ and $k_i^{'}$, $1 \leq i \leq n$ represents the newly introduced vertices between the edges $m_i$ and $k_i$ respectively. By the definition of the Total graph $V\left[T\left(T_{m,n}\right)\right] = V\left(T_{m,n}\right)\ \cup\ E\left(T_{m,n}\right) = \{x_i : 1 \leq i \leq n\}$ $\cup \{y_i : 1 \leq i \leq n\} \cup \left\{m_i^{'} : 1 \leq i \leq n\right\} \cup \left\{k_i^{'} : 1 \leq i \leq n\right\}$.
Assign a proper 5 coloring to $V\left[T\left(T_{m,n}\right)\right]$ as follows:
**Case 1:** If $m \equiv 0\,(mod\ 3)$,$n \geq 1$
For $1 \leq i \leq n$, assign colors $3, 4, 2, 3, 4, 2, ..., 3, 4, 2$ to consecutive vertices of $m_i^{'}$, colors $1, 2, 3, 1, 2, 3, ..., 1, 2, 3$ to consecutive vertices of $x_i$, colors $5, 3, 5, 3, ..., 5, 3$ to consecutive vertices of $k_i^{'}$, colors $4, 2, 4, 2, ..., 4, 2$ to consecutive vertices of $y_i$. This shows that this is a maximum b-coloring with 5 colors.
**Case 2:** If $m \equiv 1\,(mod\ 3)$
For $1 \leq i \leq n$, assign colors $2, 4, 1, 4, 2, 4, 1, 4, ..., 2, 4, 1, 4$ to consecutive vertices of $m_i^{'}$, colors $1, 3, 5, 2, 1, 3, 5, 2, ..., 1, 3, 5,$ $2$ to consecutive vertices of $x_i$, colors $5, 1, 2, 5, 1, 2, ..., 5, 1, 2$ to consecutive vertices of $k_i^{'}$, colors $3, 5, 3, 5, ...,$ $3, 5$ to consecutive vertices of $y_i$. This shows that this is a maximum b-coloring with 5 colors.
**Case 3:** If $m \equiv 2\,(mod\ 3)$
For $1 \leq i \leq n$, assign colors $2, 3, 5, 3, 4, 2, 3, 5, 3, 4..., 2, 3, 5, 3,$ $4$ to consecutive vertices of $m_i^{'}$, colors $1, 4, 1, 2, 5, 1, 4, 1, 2, 5...,$ $1, 4, 1, 2, 5$ to consecutive vertices of $x_i$, colors $5, 4, 1, 5, 4, 1...,$ $5, 4, 1$ to consecutive vertices of $k_i^{'}$, colors $1, 2, 3, 1, 2, 3..., 1, 2,$ $3$ to consecutive vertices of $y_i$. This shows that this is a maximum b-coloring with 5 colors.
From the above cases it follow that $\varphi\left[T\left(T_{m,n}\right)\right] = 5$.

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