An Analytical Solution of Non Linear One Dimensional Diffusion Equation by Homotopy Perturbation Method

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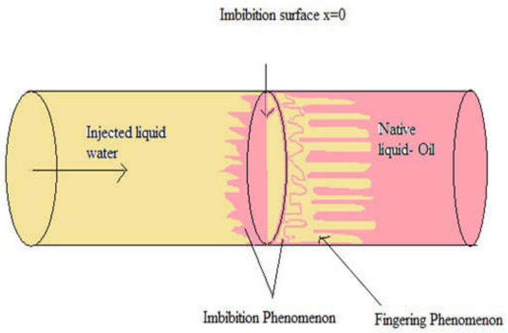
Abstract: The present paper discusses analytically the phenomenon of imbibition in double phase flow through homogenous porous media by using Homotopy perturbation method. The basic assumptions underlying in the present investigation is that the oil and water form two immiscible liquidphases and the latter represents preferentially wetting phase. . The Saturation of injected water is calculated by Homotopy perturbation method for Nonlinear differential equation of Imbibition phenomena under assumption that Saturation is decomposed in to saturation of different levels. The obtained results as compared with previous works are highly accurate. Also Homotopy perturbation method provides continuous solution in contrast to finite difference method, which only provides discrete approximations. He's Homotopy Perturbation method is powerful and capable method to solve liner and nonlinear equation directly. Graphical illustration has been done by mat lab.

Key words: *Imbibition Phenomenon, Non-linear diffusion equation, HomotopyPerturbation method.*

INTRODUCTION

This paper discusses mathematically phenomenon of imbibition in double phase flow of two immiscible fluids in a homogeneous porous media with capillary pressure.it is well known that when a porous medium of length (L), field with some fluid (N) is brought in to contact with another fluid (I) preferentially wets the medium, it is observe that there is a spontaneous flow of the wetting fluid into the medium and a counter of the resident fluid from the medium. The phenomenon is called Imbibition and has been discussed by many authors from different points.[6][8][10]

One of the most important process in oil recovery is the spontaneous imbibition which is driven by capillary force. Such spontaneous imbibition may occur in the form of co-current imbibition or counter current imbibition. The direction of flow is the main difference between these two crucial mechanisms for imbibition. In co-current imbibition, the wetting and non-wetting phases flow in the same direction with the non-wetting phase being pushed out ahead of the wetting phase. In counter current imbibition, the wetting and non-wetting phases flow in the opposite directions.[8] In the present paper, we have discussed the imbibition phenomenon arising in the flow of two immiscible fluid flows through homogeneous porous media with the effect of capillary pressure and obtained an approximate solution of the nonlinear differential system governing imbibition phenomena through Homotopy perturbation method.



FORMULATION OF MATHEMATICAL PROBLEM:

Water is injected at common interface in homogeneous porous medium connecting oil which will displaced by injecting water.

Hence water and oil both will satisfy Darcy's law given by Bear [1] which gives velocity of water and oil respectively as,

$$V_w = -\frac{K_w}{\mu_w} K \left(\frac{\partial P_w}{\partial x}\right) \qquad (1.1)$$

$$V_o = -\frac{K_o}{\mu_o} K \left(\frac{\partial P_o}{\partial x}\right) \qquad (1.2)$$

Where,

K = Permeability of homogenous porous medium

μ_o = The constant kinematic viscosities of oil

μ_w = The constant kinematic viscosities of water

P_o = Pressures of oil

P_w = Pressures of water

K_o = Relative permeability of oil

K_w = Relative permeability of water

The flow is counter current so for the Imbibition Phenomenon,

$$V_w = -V_o \qquad (1.3)$$

There for from (1.1) and (1.2) we can write,

$$\frac{K_w}{\mu_w}\frac{\partial P_w}{\partial x} + \frac{K_o}{\mu_o}\frac{\partial P_o}{\partial x} = 0 \qquad (1.4)$$

The definition of capillary pressure P_c gives,

$$P_c = P_o - P_w \qquad (1.5)$$

Combining equation (1.4) and (1.5), we get

$$\left(\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}\right)\frac{\partial P_w}{\partial x} + \frac{K_o}{\mu_o}\frac{\partial P_c}{\partial x} = 0 \qquad (1.6)$$

Substituting the value of $\frac{\partial P_w}{\partial x}$ from the equation (1.6) in to equation (1.1) , we obtain

$$V_w = \frac{K\left(\frac{K_w}{\mu_w}\right)\left(\frac{K_o}{\mu_o}\right)}{\left(\frac{K_w}{\mu_w} + \frac{K_o}{\mu_o}\right)} \frac{\partial P_c}{\partial x} \qquad (1.7)$$

Since water and oil are following in a porous medium thorough interconnected capillaries during the phenomenon of imbibition due to capillary pressure of water and oil.

The equation of continuity for the water is,

$$P \frac{\partial S_w}{\partial t} + \frac{\partial V_w}{\partial x} = 0 \qquad (1.8)$$

Where P is the porosity of the medium and S_w is the saturation of water.

Substituting the value of V_w from equation (1.7) in to equation (1.8), it becomes

$$P \frac{\partial S_w}{\partial t} + \frac{\partial}{\partial x}\left[K \frac{K_w K_o}{K_w\mu_o + K_o\mu_w} \frac{dP_c}{dS_w}\frac{\partial S_w}{\partial x}\right] = 0 \qquad (1.9)$$

This equation (1.9) is a non-linear partial differential equation, which describes the linear counter current imbibition phenomenon of two immiscible fluids flow thorough homogeneous cylindrical medium.

It is well know that, relative permeability is the function of displacing fluid saturation. Then at this stage for the mathematical analysis, we assume standard forms of (schidegger and Jonson) for the analytical relationship between the relative permeability, phase saturation and capillary pressure phase saturation as,

$K_w = S_w$,

$K_o = S_o = (1 - \alpha S_w)$, where$\alpha = 1.11$

$P_c = -\beta S_w$ (1.10)

Since the present investigation involves water and oil ,there for according to Scheidegger [1960] we have,

$$\frac{K_w K_o}{K_w\mu_o + K_o\mu_w} \approx \frac{K_o}{\mu_o} = \frac{1 - \alpha S_w}{\mu_o} = \frac{S_w}{\mu_o} \qquad (1.11)$$

Where P is the porosity of medium , K is permeability of medium, s_w and s_o are saturation of fluids water and oil respectively, P_c is capillary pressure , β is capillary pressure coefficient.

Using the relation equation (1.9) became,

$$\frac{\partial S_w}{\partial t} = \frac{K\beta}{\mu_0 P}\frac{\partial}{\partial x}\left[S_w \frac{\partial S_w}{\partial x}\right] \qquad (1.12)$$

Now we choose new variables converting (1.12) into dimensional less form,

$\mathrm{X} = \frac{x}{L}$ and $T = \frac{K\beta}{\mu_0 P L^2} t$

Equation (1.12) becomes

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[S_w \frac{\partial S_w}{\partial X}\right] \text{ [7]} \qquad (1.13)$$

PROBLEM: I

The nonlinear diffusion equation is

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[S_w \frac{\partial S_w}{\partial X}\right] \qquad (2.1)$$

The boundary conditions are ,

$S_w(0,T) = T$ & $S_w(1, T) = 1 + T$ (2.2)

Initial condition is given by

$S_w(X,0) = X$ (2.3)

Homotopy $v(r,p)\colon \Omega \times [0,1] \to R$ for equation (2.1) is define as

$$\mathrm{H(v,p)} = (1-p)\left[\frac{\partial v}{\partial T} - \frac{\partial v_o}{\partial T}\right] + p\left[\frac{\partial v}{\partial T} - v\frac{\partial^2 v}{\partial T^2} - \left(\frac{\partial v}{\partial T}\right)^2\right] = 0$$

$$= (1-p)\left[\frac{\partial v_o}{\partial T} + p\frac{\partial v_1}{\partial T} + p^2\frac{\partial v_2}{\partial T} + p^3\frac{\partial v_3}{\partial T} + p^4\frac{\partial v_4}{\partial T} + \cdots - \frac{\partial v_o}{\partial T}\right]$$

$$+ p\left(\begin{array}{c}\left[\frac{\partial v_o}{\partial T} + p\frac{\partial v_1}{\partial T} + p^2\frac{\partial v_2}{\partial T} + p^3\frac{\partial v_3}{\partial T} + p^4\frac{\partial v_4}{\partial T} + \cdots\ldots\right] \\ -(v_0 + pv_1 + p^2 v_2 + p^3 v_3 + p^4 v_4 + \cdots) \\ \left(\frac{\partial^2 v_0}{\partial X^2} + p\frac{\partial^2 v_1}{\partial X^2} + p^2\frac{\partial^2 v_2}{\partial X^2} + p^3\frac{\partial^2 v_3}{\partial X^2} + p^4\frac{\partial^2 v_4}{\partial X^2} + \cdots\right) \\ -\left(\frac{\partial v_o}{\partial X} + p\frac{\partial v_1}{\partial X} + p^2\frac{\partial v_2}{\partial X} + p^3\frac{\partial v_3}{\partial X} + p^4\frac{\partial v_3}{\partial X} + \cdots\ldots\right)^2\end{array}\right)$$

(2.4)

Comparing powers of P ,

$$p^0 : \frac{\partial v_0}{\partial T} - \frac{\partial v_0}{\partial T} = 0$$

$$p^1 : \frac{\partial v_1}{\partial T} + \frac{\partial v_0}{\partial T} - v_0\left(\frac{\partial^2 v_0}{\partial X^2}\right) + \left(\frac{\partial v_0}{\partial X}\right)^2 = 0$$

$$p^2 : \frac{\partial v_2}{\partial T} - v_0\left(\frac{\partial^2 v_1}{\partial X^2}\right) - v_1\left(\frac{\partial^2 v_0}{\partial X^2}\right) + 2\left(\frac{\partial v_0}{\partial X}\right)\left(\frac{\partial v_1}{\partial X}\right) = 0$$

$$p^3 : \frac{\partial v_3}{\partial T} - v_0\left(\frac{\partial^2 v_2}{\partial X^2}\right) - v_1\left(\frac{\partial^2 v_1}{\partial X^2}\right) - v_2\left(\frac{\partial^2 v_0}{\partial X^2}\right) + \left(\frac{\partial v_1}{\partial X}\right)^2 + 2\left(\frac{\partial v_0}{\partial X}\right)\left(\frac{\partial v_2}{\partial X}\right) = 0$$

$$p^4 : \frac{\partial v_4}{\partial T} - v_0\left(\frac{\partial^2 v_3}{\partial X^2}\right) - v_1\left(\frac{\partial^2 v_2}{\partial X^2}\right) - v_2\left(\frac{\partial^2 v_1}{\partial X^2}\right) + v_3\left(\frac{\partial^2 v_0}{\partial X^2}\right) + 2\left(\frac{\partial v_0}{\partial X}\right)\left(\frac{\partial v_3}{\partial X}\right) + 2\left(\frac{\partial v_1}{\partial X}\right)\left(\frac{\partial v_2}{\partial X}\right) = 0$$

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Solving all above partial differential equation we get,

$v_0 = X$

$v_1 = T$

$v_2 = 0$

$v_3 = 0$

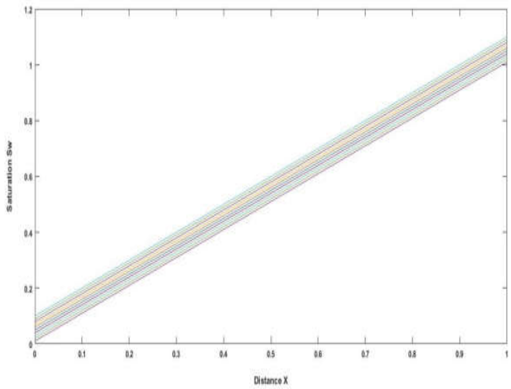
The solution of the equation (2.1) can be written as,

$$S_w(X,T) = v_0 + v_1 + v_2 + \cdots \dots\dots\dots\dots\dots \quad (2.5)$$

Putting all the values in equation (2.5) we get,

$$S_w(X,T) = X + T \quad (2.6)$$

Equation (2.6) is the exact solution of equation (2.1).



PROBLEM: II

$$\frac{\partial S_w}{\partial T} = \frac{\partial}{\partial X}\left[S_w \frac{\partial S_w}{\partial X}\right] \quad (3.1)$$

Boundary conditions are,

$$S_w(0,T) = 0 \text{ and } \quad S_w\left(\frac{\pi}{2}, T\right) = e^T \quad (3.2)$$

Initial condition is given by

$$S_w(T,0) = \sin X \quad (3.3)$$

Homotopy $v(r,p)\colon\ \Omega\ \times[0,1]\to R$ for equation (3.1) is define as

$$\mathrm{H(v,p)}=(1-p)\left[\frac{\partial v}{\partial T}-\ \frac{\partial v_o}{\partial T}\right]+p\ [\ \frac{\partial v}{\partial T}-v\ \frac{\partial^2 v}{\partial X^2}\ -\ \left(\frac{\partial v}{\partial X}\right)^2]=\ 0$$

$$=(1-p)\ [\ \frac{\partial v_o}{\partial T}\ +\ p\ \frac{\partial v_1}{\partial T}\ +\ p^2\frac{\partial v_2}{\partial T}+\ p^3\frac{\partial v_3}{\partial T}+\ p^4\frac{\partial v_3}{\partial T}+\cdots-\ \frac{\partial v_o}{\partial T}\ ]$$

$$+p\begin{pmatrix}\left[\frac{\partial v_o}{\partial T}\ +\ p\ \frac{\partial v_1}{\partial T}\ +\ p^2\frac{\partial v_2}{\partial T}+\ p^3\frac{\partial v_3}{\partial T}+\ p^4\frac{\partial v_4}{\partial T}+\cdots\ldots\right]\\ -(\ v_0+pv_1+\ p^2v_2+\ p^3v_3+p^4v_4+\cdots\ )\\ (\frac{\partial^2 v_0}{\partial X^2}+p\ \frac{\partial^2 v_1}{\partial X^2}+\ p^2\frac{\partial^2 v_2}{\partial X^2}\ +\ p^3\frac{\partial^2 v_3}{\partial X^2}\ +\ p^4\frac{\partial^2 v_4}{\partial X^2}+\cdots)\\ -\left(\frac{\partial v_o}{\partial X}\ +\ p\ \frac{\partial v_1}{\partial X}\ +\ p^2\frac{\partial v_2}{\partial X}+\ p^3\frac{\partial v_3}{\partial X}+\ p^4\frac{\partial v_4}{\partial X}+\cdots\ldots\right)^2\end{pmatrix}=0$$

(3.4)

Comparing the coefficient of like powers of p ,The following approximation are obtained by,

$$p^0:\ \frac{\partial v_0}{\partial T}\ -\ \frac{\partial v_0}{\partial T}\ =0$$

$$p^1:\ \ \frac{\partial v_1}{\partial T}\ -\ \frac{\partial v_0}{\partial T}+\frac{\partial v_0}{\partial T}+\frac{\partial v_0}{\partial T}+\ \left(\frac{\partial^2 v_0}{\partial X^2}\right)=0$$

$$p^2:\ \ \frac{\partial v_2}{\partial T}-\frac{\partial v_1}{\partial T}+\frac{\partial v_1}{\partial T}+\frac{\partial^2 v_1}{\partial X^2}=0$$

$$p^3:\ \ \frac{\partial v_3}{\partial T}-\ v_0\ \left(\frac{\partial^2 v_2}{\partial T^2}\right)\ -\ v_1\left(\frac{\partial^2 v_1}{\partial X^2}\right)-v_2\left(\frac{\partial^2 v_0}{\partial X^2}\right)+\left(\frac{\partial v_1}{\partial T}\right)^2+2\ \left(\frac{\partial v_0}{\partial X}\right)\left(\frac{\partial v_2}{\partial X}\right)\ =0$$

$$p^4\!:\ \frac{\partial v_4}{\partial T}-\ v_0\ \left(\frac{\partial^2 v_3}{\partial T^2}\right)-\ v_1\left(\frac{\partial^2 v_2}{\partial X^2}\right)-v_2\left(\frac{\partial^2 v_1}{\partial T^2}\right)+\ v_3\left(\frac{\partial^2 v_0}{\partial X^2}\right)+2\ \left(\frac{\partial v_0}{\partial X}\right)\left(\frac{\partial v_3}{\partial X}\right)\ +2\left(\frac{\partial v_1}{\partial X}\right)\left(\frac{\partial v_2}{\partial X}\right)=0$$

....

Solving all above partial differential equation we get,

$v_0=\sin X$

$v_1=\ T\ sinX$

$v_2=\dfrac{T^2}{2!}\sin X$

$v_3=\ \dfrac{T^3}{3!}\sin X$

$v_4=\dfrac{T^4}{4!}\ \sin X$

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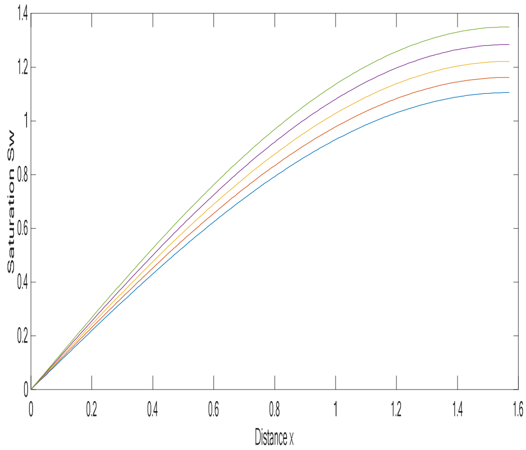
Solution of equation (3.1) can be written as ,

$S_w(X,T)\ =\ \ v_0+v_1+v_2+\cdots\ldots\ldots\ldots\ldots\ldots..$ (3.5)

Putting all the values in equation (3.5) we get,

$S_w\ (X,T)=\sin X\ +T\sin X+\ \frac{T^2}{2!}\sin X+\ \frac{T^3}{3!}\sin X+\ \frac{T^4}{4!}\ \sin X+\ \ldots\ldots\ldots\ldots$ (3.6)

Equation (3.6) is the solution of equation (3.1).



CONCLUSION

An oil and water imbibition problem in a homogenous porous medium has been analytically discussed under special conditions, by using Homotopy perturbation method.The nonlinear diffusion equation are solved by Homotopy perturbation method. The results for the saturation of water are shown in Graphs. It can be seen that the saturation of water increase linearly with time and the space variable in first case and the saturation is increase with time and space in second case also. The results shows that in both the cases the water saturation is increased with time and spaces properly.

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