Painleve Analysis and Symmetry Analysis of the Two Dimensional Hirota – Satsuma Equation

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Abstract: We discuss the painleve analysis of the Hirota – Satsuma equation. Next discuss the symmetries of the Hirota – Satsuma equation. We classify one-dimensional subalgebras of the Hirota – Satsuma equation. Further reduction of these equations to second order equations. Finally find out the numerical solution of this equation.

1. INTRODUCTION

The solitary wave and soliton phenomenon was first described in 1834 by John Scott Russell (1808–1882) when he followed the path of a solitary wave in the Union Canal in Scotland [1]. Soliton is not defined in a unique way. Solutions of nonlinear wave equations which have the following three properties are called soliton.

1- It is confined in a finite region of space.
2- Their shape and velocity are not changed.
3- After the collision with other solitons, its shape is preserved.

Answers that include the second property, have been called Solitary waves. The inherent stability of solitons, enable them to be sent over long distances without the use of repeaters and could potentially double transmission capacity. Soliton waves are quite stable, and in case of disturbance continue to move to its initial state. After Russell, more than a century, the solitons were not needed. Then in 1965, Norman Zabvsky, from Bell lab and Martin Kruskal of Princeton university, who described the behavior of solitons in terms in mathematical expression. Since then, gradually, solitons were used not only to describe water waves, but also in other fields of physics that deal with the wave and showed excellent performance. [2, 3].

Nonlinear partial differential equations (NPDE) in different scientific fields such as fluid mechanics, solid state physics, plasma physics, optical fibers, chemical physics [4, 5] , and so on have the most importance subjects for study. Finding exact responses to these equations will help us to better understanding of our environmental nonlinear physical phenomena. For most non-linear partial differential equations, soliton solutions can be defined. One of these equations is nonlinear equation of generalized Hirota -Satsuma coupled with a KdV system which will be shown below:

$$u_t = \frac{1}{4}\, u_{xxx} + 3\, u\, u_x + 3(-v^2 + w)_x\,, \qquad v_t = -\frac{1}{2}\, v_{xxx} - 3\, u\, v_x\,, \quad w_t = -\frac{1}{2}\, w_{xxx} - 3\, u\, w_x$$

Mayil Vaganan and Muthumari [6] reported invariant solutions of another KdVE using Lie's group of infinitesimal transformations [7], [8]. In 1969, Zabolotskaya and Khokhlove derived the ZK equation$(u_t + u\, u_x - \beta\, u_{xx})_x + \gamma u_{yy} = 0$. Yet another model equation is derived by Kadomtsev and Petviashvili [9] $(u_t + u\, u_x + \epsilon\, u_{xx})_x + \lambda u_{yy} = 0, \lambda = \pm 1$ is the generalization of two spatial dimensions, x and y, of the KdV equation. But David, Levi and Winternitz [10] generalized KP equation to describe water waves in straits or rivers.

Muthumari [11], discuss about Painleve Analysis and Symmetry Analysis of the Two Dimensional Variable Coefficient Burgers Equation, Painleve property of KdV equation with nonunifromities in Brugarino T[12]. In 1985, Painleve property of a Space-Dependent Burger's Equation by W. H. STEEB and W.STRAMPP [13]. Painleve analysis and reducibility to the canonical form for the generalized KP equation by Tommaso Brugarino and Antaonio M. Greaco [14]. Discuss Painleve analysis and Backlund transformation in the KS equation in Robert Conte and Micheline Musette [15].

In this paper, we discuss Painleve property and symmetries of the Hirota – Satsuma equation

$$u_t = \frac{1}{2} u_{xxx} + 3\, u\, u_x - 6\, w\, w_x$$
$$w_t + w_{xxx} + 3\, u\, w_x = 0 \qquad (1.1)$$

This paper is organised as follows: In section 2, General view of Painleve analysis. In section 3, Painleve analysis of the Hirota – Satsuma equation. In section 4, we perform a symmetry classification of the equation using Lie classical method. In section 5, using one-dimensional subalgebras we reduce the given PDE and we find out the numerical solution of the reduced equation. In section 6, we summarize the results of the present work.

2. Painleve analysis: generalities

An essential question in the study of NPDE is the nature of the singularities of the solutions (poles, branch points or essential singularities) and their position (fixed or movable).

For this purpose, the Painleve analysis, which has been renewed by Ablowitz et al [16] for ordinary differential equations (ODE). It consists in looking for the general solution of the PDE in the form (written here in the case of one dependent and two independent variables):

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)[\emptyset(x,t)]^{n+k} \qquad (2.1)$$

where k is negative, $\emptyset(x,t) = 0$ is the equation of a non-characteristic $(\emptyset_x \emptyset_t \neq 0)$ singular manifold, and the functions u_n have to be determined by substitution of expansion (2.1) in the PDE, which becomes:

$$\sum_{n=0}^{\infty} E_n(u_0, \cdots, u_n, \emptyset)[\emptyset]^{n+q} = 0 \qquad (2.2)$$

where q is some negative constant. E_n depends on $\emptyset$ only by the derivatives of $\emptyset$.

The successive practical steps of Painleve analysis are the following.

(i) Determine the possible leading orders k by balancing two or more terms of the PDE and expressing that they dominate the other terms.

(ii) Solve equation $E_0 = 0$ for non-zero values of u_0; this may lead to several solutions, called branches.

(iii) Find the resonances, i.e. the values of n for which u_n cannot be determined from equation $E_n = 0$. This last equation has usually the form:

$$\forall n > 0 \qquad E_n \equiv (n+1)P(n)\emptyset_x^j \emptyset_t^{m-j} u_n + Q(u_0, \cdots, u_{n-1}, \emptyset) = 0 \qquad (2.3)$$

where m is the order of the PDE, $0 \leq j \leq m$ and P a polynomial of degree $m-1$. The values of the resonances are the zeros of P.

(iv) Determine if the resonances are 'compatible' or not. At a resonance, after substitution in (2.3) of the previously computed u_l, $l \leq n-1$, the function Q is either zero, in which case u_n can be arbitrarily chosen and the resonance is said to be compatible, or non-zero and the expansion (2.1) does not exist for arbitrary $\emptyset$. The Painleve property is characterised by the fact that k is a negative integer and all resonances occur at positive integer values of n and are compatible.

3. Painleve analysis of (1.1)

We are looking for a solution of (1.1) in the Laurent series expansion

$$u(x,\ t) = \sum_{n=0}^{\infty} u_n(x,\ t)[\emptyset(x,t))]^{n+k} \quad \text{and} \quad w(x,\ t) = \sum_{n=0}^{\infty} w_n(x,\ t)[\emptyset(x,t))]^{n+k} \qquad (3.1)$$

where $u_n(x,\ t)$ and $w_n(x,\ t)$ are analytic functions in a neighbourhood of the singular manifold $\emptyset(x,t) = 0$ and k is an integer to be determined. Inserting the ansatz

$$u(x,t) \cong u_0 \emptyset^k \quad \text{and} \quad w(x,t) \cong w_0 \emptyset^k$$

In (1.1) and comparing the exponents, we find that the leading-order analysis gives the value $k = -2$. Insertng the ansatz (3.1) together with $k = -2$ in (1.1), then we find the recursion relations for $u_n(x,\ t)$ and $w_n(x,\ t)$

$$-u_{n-3,t} - u_{n-2}(n-4)\emptyset_t + \frac{1}{2}u_{n-3,xxx} + \frac{3}{2}u_{n-2,xx}(n-4)\emptyset_x + \frac{3}{2}u_{n-1,x}(n-3)(n-4){\emptyset_x}^2 + \frac{3}{2}u_{n-2,x}(n-4)\emptyset_{xx}$$
$$+\frac{1}{2}u_n(n-2)(n-3)\,(n-4){\emptyset_x}^3 + \frac{3}{2}u_{n-1}(n-3)(n-4)\emptyset_x\,\emptyset_{xx} + \frac{1}{2}u_{n-2}\,(n-4)\emptyset_{xxx} + 3\sum_{m=0}^{n} u_{m-1}\,u_{n-m,x}$$
$$+3\sum_{m=0}^{n} u_m\,u_{n-m}(n-m-2)\,\emptyset_x - 6\sum_{m=0}^{n} w_{m-1}\,w_{n-m,x} - 6\sum_{m=0}^{n} w_m\,w_{n-m}\,(n-m-2)\emptyset_x = 0$$
$$n = 0,1,2,\ldots \tag{3.2}$$

$$w_{n-3,t} + w_{n-2}(n-4)\emptyset_t + w_{n-3,xxx} + 3\,w_{n-2,xx}(n-4)\emptyset_x + 3\,w_{n-1,x}(n-3)(n-4){\emptyset_x}^2 + 3\,w_{n-2,x}(n-4)\emptyset_{xx}$$
$$+w_n(n-2)(n-3)\,(n-4){\emptyset_x}^3 + 3\,w_{n-1}(n-3)(n-4)\emptyset_x\,\emptyset_{xx} + w_{n-2}\,(n-4)\emptyset_{xxx} + 3\sum_{m=0}^{n} u_{m-1}\,w_{n-m,x}$$
$$+3\sum_{m=0}^{n} u_m\,w_{n-m}(n-m-2)\,\emptyset_x = 0 \qquad n = 0,1,2,\ldots \tag{3.3}$$

In collecting terms involving u_n and w_n, it is found that

$$\frac{1}{2}(n-2)(n-3)\,(n-4)\emptyset_x^3\,u_n - 12\,(n-2)\emptyset_x^3 u_n - 12\,(n-2)\emptyset_x^3\,w_n = F(x,\ t,u_{n-1},\ldots,u_0,w_{n-1},\ldots,w_0,\emptyset_t,\emptyset_x)$$
$$(n-2)(n-3)\,(n-4)\emptyset_x^3\,w_n - 12\,(n-2)\emptyset_x^3\,w_n = F(x,\ t,u_{n-1},\ldots,u_0,w_{n-1},\ldots,w_0,\emptyset_t,\emptyset_x) \tag{3.4}$$

Equation (3.4) determines the coefficients u_n and w_n of the series expansion (3.1), provided that $n \neq 0,2,7$. These values of n are called the "resonance" of the recursion relations and allow the introduction of arbitrary functions $u_2,\ u_7,\ w_2$ and w_7 For $n = -2$, the series (3.1) is undefined and therefore the resonance at $n = -2$ corresponds to "arbitrary" function $\emptyset$ defining the singular manifold.

Put $n = 0$ in (3.2), we get $u_0 = -4\,{\emptyset_x}^2$ and $w_0 = 2\,{\emptyset_x}^2$ (3.5)

Put $n\ =\ 2,7$ in (3.2) and (3.3) using (3.4), we get the following system of equations

$$-8\,{\emptyset_x}^2\emptyset_t + 24\,{\emptyset_{xx}}^3 + 24\,\emptyset_x\,\emptyset_{xx}\,\emptyset_{xxx} + 24\,\emptyset_x{\emptyset_{xx}}^2 + 4\,{\emptyset_x}^2\,\emptyset_{xxx} + 24\,u_2\,{\emptyset_x}^3 + 24\,w_2\,{\emptyset_x}^3 = 0$$
$$-4\,\emptyset_x^2\,\emptyset_t - 48\,\emptyset_x{\emptyset_{xx}}^2 - 28\,{\emptyset_x}^2\,\emptyset_{xxx} - 12\,u_2\,\emptyset_x^3 = 0$$
$$-6\,u_7\,{\emptyset_x}^3 - 36\,w_7\,{\emptyset_x}^3 = 0$$
$$60\,w_7\,{\emptyset_x}^3 - 60\,w_7\,{\emptyset_x}^3 - 12u_7\,{\emptyset_x}^3 = 0$$

Furthermore, if we get the arbitrary functions u_7 and w_7 equal to zero. Then $u_n = 0$ for $n \geq 3$ and $w_n = 0$ for $n \geq 3$ and u_2 and w_2 are solutions of the equation (1.1) $u_{2,t} = \frac{1}{2}\,u_{2,xxx} + 3\,u_2\,u_{2,x} - 6\,w_2\,w_{2,x}$, $w_{2,t} + w_{2,xxx} + 3\,u_2\,w_{2,x} = 0$.

4. The symmetry group and its Lie algebra

We consider the one – parameter Lie group of infinitesimal transformations (Olver [3], Blumen and Kumei [18]) in x,t and u given by $x_i^* = x_i + \epsilon\,\xi_i(x,t;u,w) + O(\epsilon^2)$, i = 1, 2, 3

where $x_1 = x,\ \ x_2 = t,\ \ x_3 = u,\ x_4 = w,\ \ \xi_1 = X(x,t;u,w),\ \ \xi_2 = T(x,t;u,w), \xi_3 = U(x,t;u,w),\ \xi_4 = W(x,t;u,w)$ and the corresponding vector field $V = X(x,t;u,w)\partial_x + T(x,t;u,w)\,\partial_t + U(x,t;u,w)\partial_u + W(x,t;u,w)\partial_w$ then the third prolongation $pr^{(3)}V$ of the corresponding above vector field

$$pr^{(3)}V = V + U^x\frac{\partial}{\partial u_x} + U^t\frac{\partial}{\partial u_t} + U^{xx}\frac{\partial}{\partial u_{xx}} + U^{xt}\frac{\partial}{\partial u_{xt}} + U^{tt}\frac{\partial}{\partial u_{tt}} + U^{xxx}\frac{\partial}{\partial u_{xxx}} + U^{xxt}\frac{\partial}{\partial u_{xxt}}$$
$$+ U^{ttx}\frac{\partial}{\partial u_{ttx}} + U^{ttt}\frac{\partial}{\partial u_{ttt}} + W^x\frac{\partial}{\partial w_x} + W^t\frac{\partial}{\partial w_t} + W^{xx}\frac{\partial}{\partial w_{xx}} + W^{xt}\frac{\partial}{\partial w_{xt}} + W^{tt}\frac{\partial}{\partial w_{tt}} + W^{xxx}\frac{\partial}{\partial w_{xxx}}$$
$$+ W^{xxt}\frac{\partial}{\partial w_{xxt}} + W^{ttx}\frac{\partial}{\partial w_{ttx}} + W^{ttt}\frac{\partial}{\partial w_{ttt}}$$

Equation (1.1) can be written as

$\Omega_1 \equiv u_t - \frac{1}{2} u_{xxx} - 3\,u\,u_x + 6\,w\,w_x$ and $\Omega_2 \equiv w_t + w_{xxx} + 3\,u\,w_x$

the vector field satisfies $pr^{(3)}V\ \Omega_1(x,\ t;\ u,w)|_{\Omega_1(x,t;u,w)=0} = 0$ and $pr^{(3)}V\ \Omega_2(x,\ t;\ u,w)|_{\Omega_2(x,t;u,w)=0} = 0$ using the above equation, we get the infinitesimals X, Y, T and U are

$$X = k_3 + k_1 t + k_4\,x\,, \quad T = k_2 + 3\,k_4\,t, \quad U = \frac{1}{3}(-k_1 - 6\,k_4 u), \quad W = -2\,k_4\,w$$

Now we write down the four symmetry generators corresponding to each of the constants $k_i,\ i = 1,2,3,4$ involved in the infinitesimals, viz.,

$$V_1 = t\,\partial x, \quad V_2 = \partial t\,, \quad V3 = \partial x, \quad V4 = x\,\partial x + 3t\,\partial t - 2u\,\partial u - 2w\,\partial w$$

These symmetry generators form a closed Lie algebra as is seen from the following

Commutator Table

[,]	V_1	V_2	V_3	V_4
V_1	*0*	V_3	*0*	$2V_1$
V_2	$-V_3$	*0*	*0*	$-3V_2$
V_3	*0*	*0*	*0*	$-V_3$
V_4	$-2V_1$	$3V_2$	V_3	*0*

Where [,] stands for the Lie bracket.

5. Symmetry reductions of (1.1) by one-dimensional subalgebras

We consider the reductions of (1.1) under each generator separately.

Case 1: Subalgebra $L_{s,1} = \{V_1\}$

The characteristic equation associated the generator V_1 is

$$\frac{dx}{t} = \frac{dt}{0} = \frac{du}{0} = \frac{dw}{0} \quad (5.1)$$

Integration of (5.1) yields the similarity transformation

$$u = P(\xi);\ \ w = Q(\xi);\ \ \xi = t \quad (5.2)$$

Using (5.2) in (1.1), obtain the reduced ODEs

$P'(\xi) = 0$ and $Q'(\xi) = 0$

Integrate once w.r.t. ξ, we get

$P(\xi) = C_1$ and $Q(\xi) = C_2$

where C_1 and C_2 are arbitrary constants of integration.

Case 2: Subalgebra $L_{s,2} = \{V_2\}$

The characteristic equation associated the generator V_2 is

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0} = \frac{dw}{0} \quad (5.3)$$

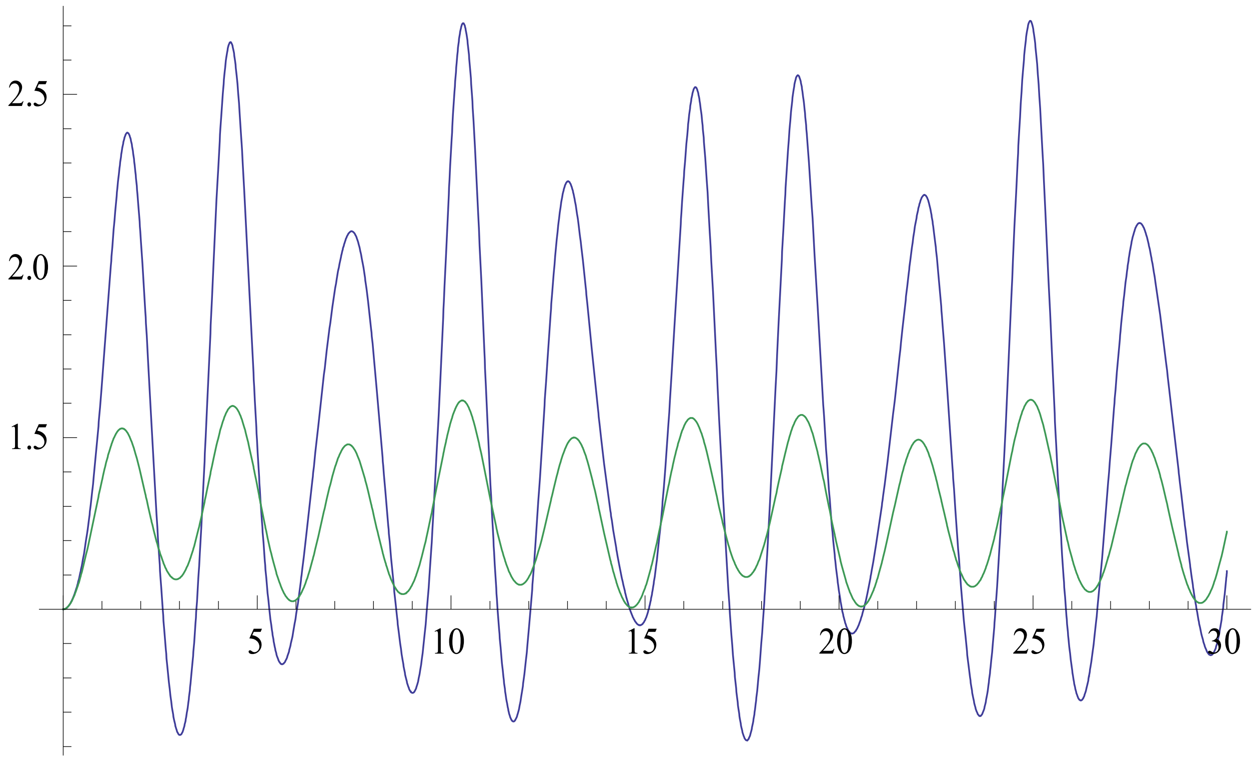
Integration of (5.3) yields the similarity transformation

$$u = P(\eta);\ w = Q(\eta);\ \eta = x \quad (5.4)$$

Using (5.4) in (1.1), obtain the reduced ODEs

$\frac{1}{2} P'''(\eta) + 3\,P(\eta)\,P'(\eta) - 6\,Q(\eta)\,Q'(\eta) = 0$ and $Q'''(\eta) + 3\,P(\eta)\,Q'(\eta) = 0$

Using Mathematica, to find the numerical solution of the above equations and plot it.



Case 3: Subalgebra $L_{s,3} = \{V_3\}$

The characteristic equation associated the generator V_3 is

$$\frac{dx}{1} = \frac{dt}{0} = \frac{du}{0} = \frac{dw}{0} \tag{5.5}$$

Integration of (5.5) yields the similarity transformation

$$u = P(\xi); \;\; w = Q(\xi); \;\; \xi = t \tag{5.6}$$

Using (5.6) in (1.1), obtain the reduced ODEs

$P'(\xi) = 0$ and $Q'(\xi) = 0$

Integrate once w.r.t. ξ, we get

$P(\xi) = C_1$ and $Q(\xi) = C_2$

where C_1 and C_2 are arbitrary constants of integration.

Case 4: Subalgebra $L_{s,4} = \{V_4\}$

The characteristic equation associated the generator V_4 is

$$\frac{dx}{x} = \frac{dt}{3t} = \frac{du}{-2u} = \frac{dw}{-2w} \tag{5.7}$$

Integration of (5.7) yields the similarity transformation

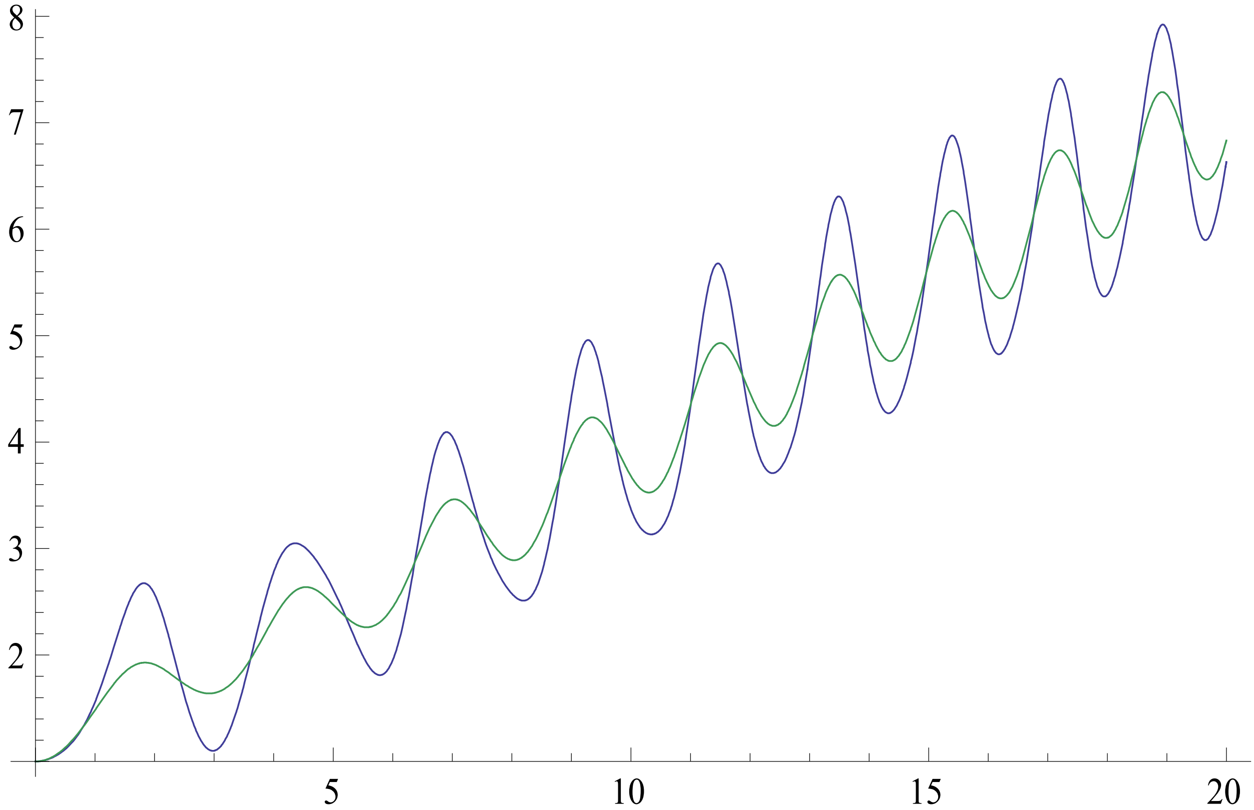
$$u = t^{-\frac{2}{3}} P(\xi); \;\; w = t^{-\frac{2}{3}} Q(\xi); \;\; \xi = x\, t^{-\frac{1}{3}} \tag{5.8}$$

Using (5.8) in (1.1), obtain the reduced ODEs

$4\,P(\xi) + 2\,\xi\,P'(\xi) + 3\,P'''(\xi) + 18\,P(\xi)P'(\xi) - 36\,Q(\xi)Q'(\xi) = 0$ and

$2\,Q(\xi) + \xi\,Q'(\xi) - 3\,Q'''(\xi) - 9\,P(\xi)Q'(\xi) = 0$

Using Mathematica, to find the numerical solution of the above equations and plot it.



6. Conclusion

The results can be summarized as follows:

- We summarize the generalzation of painleve property.
- We analyze the painleve property of Hirota – Satsuma equation.
- We determined the symmetry generators of the Hirota – Satsuma equation using Lie classical method.
- We found a classification of the one-dimensional subalgebras of the symmetry algebra under the adjoint (conjugate) action of the symmetry group.
- We plot the numerical solution of the reduced equation.

References

[1] Russell Scott J., Report of the fourteenth meeting of the British Association for the Advancement of Science, York, September (1844), 311-390, (London1845).

[2] Matveev, V.B., Salle, M. A.: Darboux Transformations and Solitons. Springer, Berlin (1991).

*[3] Forozani, Gh., Ghorveei Nosrat, M.: Solitary solution of modified bad and good Boussinesq equation by using of tanh and extended tanh methods, Indian J. Pure Appl.Math., **44,** 497-510, (2013).*

[4] Xia, T. C., Li, B., Zhang, H. Q.: New explicit and exact solutions for the Nizhnik–Novikov–Vesselov equation, Appl. Math. E-Notes, 1(2001), 139–142.

*[5] Forozani, Gh., Ghorveei Nosrat, M.: The tanh method for kink solution of some modified nonlinear equation. J. Part. Diff. Eq., **27,** 136-142 (2014).*

[6] B. Mayil Vaganan1, N. Muthumari, Symmetry classifications and reductions of $\left(u_t + u^n u_x + \alpha\, u_{xxx} + \frac{j\,u}{2\,t}\right)_x + \beta\, u_{yy} = 0$ *, IJAM, Vol. 24, No.5 2011, 635 – 662.*

[7] P. J. Olver, Applications of Lie Groups to Differential Equations, Grad. Texts in Math., 107, Springer - Verlag, New York, 1986.

[8] Bluman.G.W and Kumei.S., Symmetries and Differential Equations, (Berlin: Springer), (1989).

*[9]David.D, Levi.D and Winternitz.P., Solitons in shallow seas of variable depth and in marine starits, Stud. Appl. Math., **80**, 1-23, (1989).*

*[10] David.D, Kamran.N, Levi.D and Winternitz.P., Symmetry reduction for the Kadomstev-Petviashvili equation using a loop algebra, J. Math. Phys., **27**, 1225-37, (1986).*

[11] N. Muthumari, Painleve Analysis and Symmetry Analysis of the Two Dimensional Variable Coefficient Burgers Equation, Int. J. Adv in Math, Vol. 3, 2018, 8-16.

[12] T. Brugarino, Painleve property, auto-Backlund transformation, Lax pairs, and reduction to standard form for the Korteweg-De Vries equation with nonuniformities, American Institute of Physics, J. Math. Phys,Vol. 30, No. 5 1989, 1013 - 1015.

[13] W. H. STEEB, J. A. LOUW and W. STRAMPP, Lax Formulation, Painleve Property and Recursion Operator for a Space-Dependent Burgers' Equation, Prog. Theor. Phys., Vol. 75, No. 2 1986, 455 - 457.

[14] Tommaso Brugarino and Antonio M. Greco, Painleve analysis and reducibility to the canonical form for the generalized Kadomtsev-Petviashvili equation, American Institute of Physics, J. Math. Phys,Vol. 32, No. 1 1991, 69 - 74.

[15] Robert Conte and Micheline Musette, Painleve analysis and Backlund transformation in the Kuramoto-Sivashinsky equation, IOP Publshing Ltd, J. phys. A: Math., Vol. 22, 1989, 169 - 176.

[16] Ablowitz M J, Ramani A and Segur H 1980 A connection between nonlinear evolution equations and ordinary differential equations of P type, part I J. Math. Phys. 21 715-21; 1980 A connection between nonlinear evolution equations and ordinary differential equations of P type, part II J. Math. Phys. 21 1006-15.