Fractional order theory of thermal deflection to a 2D problem for a thin circular plate with instantaneous heat source

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Abstract

In this work, a quasi-static uncoupled theory of thermal deflection based on time fractional heat conduction equation is studied in a thin circular plate with internal heat source, whose lower surface is at zero temperature whereas the upper surface is insulated and subjected to constant temperature on the curved surface. Integral transform techniques including Mittag-Leffler functions are used to solve the problem. Numerical results for temperature distribution and thermal deflection are computed and represented graphically for copper material.

Keywords: Quasi-static; thermoelectricity; fractional order; integral transform; thermal deflection; Mittag-Leffler function.

1. Introduction

In the recent past, a lot of applications were found for fractional calculus in various engineering disciplines such as proportional-integral-derivative (PID) controllers, signal processing, fluid mechanics, viscoelasticity, mathematical biology, and electrochemistry. This has led to research in the area of non-integer calculus. The idea of fractional-order calculus is appealing but poses a serious challenge when one embarks on a journey to know the physical interpretations. Podlubny [1] gave the geometric interpretation of fractional integration as “Shadows on the walls” and its physical interpretation is “Shadows of the past.” The most important advantage of using fractional-order differential equations is their nonlocal property. This is more realistic, and an important reason why fractional calculus has gained popularity.

The classical theory of thermoelasticity has aroused much interest in recent times due to its nu-merous applications in engineering discipline such as nuclear reactor design, high energy particle accelerators, geothermal engineering, advanced aircraft structure design, etc. The heat conduction of classical coupled theory of thermoelasticity is parabolic in nature and hence predicts infinite speed of propagation of heat waves. Clearly, this contradicts the physical observations. Hence, several non-classical theories such as, Lord-Shulman theory [7], Green Lindsay theory [5] have been proposed, in which the Fourier law and the parabolic heat conduction equation are replaced by more complicated equations, which are hyperbolic in nature predicting finite wave propagation.

Green and Naghdi [6] developed the theory of thermoelasticity without energy dissipation. Chandrasekaraiah [4] gave reviews of thermoelasticity with second sound. Biot [3] formulated theory of coupled thermoelsticity to eliminate the paradox inherent in the classical uncoupled theory that elastic changes have no effect on temperature.

In the last decade, study on Quasi-static thermoelasticity incorporating the time fractional derivative has gained momentum. Povstenko [15-20] studied various problems on quasi static fractional order thermoelasticity. Many researchers [8-14] studied various problems on fractional order thermoelasticity. Boley and Weiner [2] studied the problems of thermal deflection of an axisymmetric heated circular plate in the case of fixed and simply supported edges. Roy choudhury [21] discussed the normal deflection of a thin clamped circular plate due to ramp type heating of a concentric circular region of the upper face. This satisfies the time-dependent heat conduction equation. Deshmukh and Khobragade [22] determined a quasi-static thermal deflection in a thin circular plate due to partially distributed and axisymmetric heat supply on the outer curved surface with the upper and lower faces at zero temperature. Deshmukh et al. [23] studied a quasi-static thermal deflection problem of a thin clamped circular plate due to heat generation. Deshmukh et al. [24] studied a quasi-static thermal deflection problem of a thin clamped hollow circular disk due to heat generation. Deshmukh et al. [25] studied inverse heat conduction problem in a semi–infinite circular plate and its thermal deflection by quasi-static approach. Warbhe et al. [26] studied the fractional order thermoelastic deflection in a thin hollow circular disk. Tripathi et al. [27] solved a fractional order thermoelastic deflection in a thin circular plate with constant temperature distribution within the context of quasi-static theory.

In this paper we have discussed the fractional order thermal deflection in a thin circular plate subjected to a constant heat on the curved surface and discuss the effect of fractional order parameter for $\alpha = 0.5$, 1, 1.5 and 2.

2. Formulation of the problem

We consider the homogeneous isotropic thin circular plate having the thickness h occupying the space D, defined by $0 \le r \le b$, $0 \le z \le h$. The lower surface of the plate is maintained at zero temperature whereas the upper surface is insulated. The temperature gradient is applied on the fixed circular boundary at $r = b$. For time $t > 0$, heat is generated within the thin circular plate at the rate $g(r,z,t) = g_{pi}\delta(r-r_1)\delta(z-z_1)\delta(t)$ with traction free surface. A mathematical model is prepared considering non-local Caputo type time fractional heat conduction equation of order α for a thin circular plate.

The time fractional heat conduction equation with the source term is as [29]

$$\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2}\right) + \frac{g(r,z,t)}{k} = \frac{1}{a}\frac{\partial^\alpha T}{\partial t^\alpha}, \quad 0 \le r \le b, 0 \le z \le h \tag{1}$$

with boundary condition

$$\frac{\partial T}{\partial r} = Q_0 \qquad \text{at} \quad r = b,\ t > 0 \tag{2}$$

$T = 0$ at $z = 0, t > 0$ (3)

$\frac{\partial T}{\partial z} = 0$ at $z = h, t > 0$ (4)

and initial conditions

$T = 0$ at $t = 0,\ 0 < \alpha < 2$ (5)

$\frac{\partial T}{\partial t} = 0$ at $t = 0, 1 < \alpha < 2$ (6)

The definition of Caputo type fractional derivative is given by

$$\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-\tau)^{n-\alpha-1} \frac{d^n f(\tau)}{d\tau^n} d\tau,\ n-1 < \alpha < n \tag{7}$$

For finding the Laplace transform, the Caputo derivative requires knowledge of the initial values of the function $f(t)$ and its integer derivatives of the order $k = 1,2,....,n-1$

$$L\left\{\frac{\partial^{\alpha} f(t)}{\partial t^{\alpha}}\right\} = s^{\alpha} f^{*}(s) - \sum_{k=0}^{n-1} f^{(k)}(0^{+}) s^{\alpha-1-k}, \quad n-1 < \alpha < n \tag{8}$$

where the asterisk denotes the Laplace transform with respect to time, s is the Laplace transform parameter.

Thermal Deflection $\omega(r,t)$

The differential equation satisfying the deflection function $\omega(r,t)$ is given as [23],

$$\nabla^4 \omega = -\frac{\nabla^2 M_T}{D(1-\nu)} \tag{9}$$

where, M_T is the thermal moment of the thin circular plate defined as

$$M_T = a_t E \int_0^h T(r,z,t) z\, dz \tag{10}$$

D is the flexural rigidity of the thin circular plate denoted as

$$D = \frac{Eh^3}{12(1-\nu^2)} \tag{11}$$

a_t, E and ν are the coefficients of the linear thermal expansion, the Young's modulus and Poisson's ratio of the thin circular plate material respectively and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{12}$$

Since, the inner and outer edge of thin circular plate is fixed and clamped,

$$\omega = 0 \text{ and } \frac{\partial \omega}{\partial r} = 0 \quad at \;\; r = b \tag{13}$$

Equations (1) to (13) constitute the mathematical formulation of the problem.

3. Solution

On applying Fourier, Hankel and Laplace transform and their inversions defined in [28, 29] to equation (1) and making the use of the transformed boundary and initial conditions (2) – (6), one obtains temperature distribution function expressed as follows

$$\begin{aligned} T(r,z,t) &= \sum_{p=1}^{\infty}\sum_{m=1}^{\infty} K(\eta_p, z) K_0(\beta_m, r) \\ &\times \left(\frac{a}{k} K_0(\eta_p, z_1) K_0(\beta_m, r_1)\, g_{pi}\left(t^{\alpha-1} E_{\alpha,\alpha}\left(-a\,(\beta_m^2+\eta_p^2)\, t^{\alpha}\right)\right) \right. \\ &\left. + \frac{ab}{k} K_0(\beta_m, \mathrm{b}) \frac{\sqrt{2}}{\eta_p h}\left[1-\cos\eta_p h\right] Q_0\left[1-E_{\alpha}\left(-a\,(\beta_m^2+\eta_p^2)\, t^{\alpha}\right)\right] \right) \end{aligned} \tag{14}$$

where,

$K(\eta_p, z) = \sqrt{\frac{2}{h}}\sin(\eta_p z)$, where $\eta_1, \eta_2, \ldots$ are the positive roots of transcendental equation $\cos(\eta_p h) = 0$, $p = 1, 2, \ldots$.

$K_0(\beta_m, r) = \frac{\sqrt{2}}{b}\frac{J_0(\beta_m r)}{J_0(\beta_m b)}$, where $\beta_1, \beta_2, \ldots$ are the positive roots of the transcendental equation $J_1(\beta_m b) = 0$.

Here $E_\alpha(.)$, $E_{\alpha,\alpha}(.)$ represent the Mittag-Leffler function.

Determination of thermal deflection $\omega(r,t)$

Using equation (14) into equation (10), one obtains

$$\begin{aligned} M_T &= -a_t E\sqrt{\frac{2}{h}}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin(\eta_p z) K_0(\beta_m, r) \\ &\times \left(\frac{a}{k} K_0(\eta_p, z_1) K_0(\beta_m, r_1)\, g_{pi}\left[t^{\alpha-1} E_{\alpha,\alpha}\left(-a\,(\beta_m^2+\eta_p^2)\, t^{\alpha}\right)\right] \right. \end{aligned}$$

$$+\frac{ab}{k}K_0(\beta_m,b)\frac{\sqrt{2}}{\eta_p h}(1-\cos\eta_p h)Q_0\left[1-E_\alpha\left(-a(\beta_m^2+\eta_p^2)\,t^\alpha\right)\right]\Bigg) \tag{15}$$

Assuming the solution of equation (9), satisfying the condition (13) as

$$\omega(r,t)=\sum_{m=1}^{\infty}C_m(t)[J_0(\beta_m r)-J_0(\beta_m b)] \tag{16}$$

where β_m' are the positive roots of the transcendental equation $J_1(\beta_m b)=0$.

It can be easily shown that

$\omega(r,t)=0$ at $r=b$

Now,

$$\frac{\partial\omega}{\partial r}=\sum_{m=1}^{\infty}C_m(t).\beta_m\,J_1(\beta_m r)$$

$$\frac{\partial\omega}{\partial r}=0 \text{ at } r=b$$

Hence the solution (16) satisfies the condition (13).

Now,

$$\nabla^4\omega=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)^2\sum_{m=1}^{\infty}C_m(t)[J_0(\beta_m r)-J_0(\beta_m b)] \tag{17}$$

Using the well known result

$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)J_0(\beta_m r)=-\beta_m^2\,J_0(\beta_m r)$ in equation (17), one obtains,

$$\nabla^4\omega=\sum_{m=1}^{\infty}C_m(t)\beta_m^4\,J_0(\beta_m r) \tag{18}$$

Also,

$$\nabla^2 M_T=\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)a_t E\sqrt{\frac{2}{h}}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin(\eta_p h)K_0(\beta_m,r)$$

$$\times\left\{\frac{a}{k}K_0(\eta_p,z_1)K_0(\beta_m,\mathrm{r}_1)\,\mathrm{g}_{pi}\left[t^{\alpha-1}E_{\alpha,\alpha}(-a(\beta_m^2+\eta_p^2))t^\alpha\right]\right.$$

$$+\frac{ab}{k}K_0\left(\beta_m,\mathrm{b}\right)\frac{\sqrt{2}}{\eta_p h}(1-\cos\eta_p h)Q_0\left[1-E_\alpha(-a(\beta_m^2+\eta_p^2))t^\alpha\right]\Bigg\}$$

On simplifying above equation, we get,

$$\nabla^2 M_T=-a_t E\sqrt{\frac{2}{h}}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin\left(\eta_p h\right)\beta_m^2 K_0\left(\beta_m,r\right)$$
$$\times\Bigg\{\frac{a}{k}K_0(\eta_p,z_1)K_0(\beta_m,\mathrm{r}_1)\,\mathrm{g}_{pi}\left[t^{\alpha-1}E_{\alpha,\alpha}(-a(\beta_m^2+\eta_p^2))t^\alpha\right]$$
$$+\frac{ab}{k}K_0\left(\beta_m,\mathrm{b}\right)\frac{\sqrt{2}}{\eta_p h}(1-\cos\eta_p h)Q_0\left[1-E_\alpha(-a(\beta_m^2+\eta_p^2))t^\alpha\right]\Bigg\} \qquad (19)$$

Substituting equations (18) and (19) into equation (9), one obtains,

$$\sum_{m=1}^{\infty}C_m(t)\beta_m^4 J_0(\beta_m,r)=\frac{a_t E}{D(1-\nu)}\sqrt{\frac{2}{h}}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin\left(\eta_p h\right)\beta_m^2 K_0\left(\beta_m,r\right)$$
$$\times\Bigg\{\frac{a}{k}K_0(\eta_p,z_1)K_0(\beta_m,\mathrm{r}_1)\,\mathrm{g}_{pi}\left[t^{\alpha-1}E_{\alpha,\alpha}(-a(\beta_m^2+\eta_p^2))t^\alpha\right]$$
$$+\frac{ab}{k}K_0\left(\beta_m,\mathrm{b}\right)\frac{\sqrt{2}}{\eta_p h}(1-\cos\eta_p h)Q_0\left[1-E_\alpha(-a(\beta_m^2+\eta_p^2))t^\alpha\right]\Bigg\}$$

$$C_m(t)=\frac{a_t E}{D(1-\nu)}\sqrt{\frac{2}{h}}\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin\left(\eta_p h\right)\frac{1}{\beta_m^2}\frac{\sqrt{2}}{b}\frac{1}{J_0(\beta_m,b)}$$
$$\times\Bigg\{\frac{a}{k}K_0(\eta_p,z_1)K_0(\beta_m,\mathrm{r}_1)\,\mathrm{g}_{pi}\left[t^{\alpha-1}E_{\alpha,\alpha}(-a(\beta_m^2+\eta_p^2))t^\alpha\right]$$
$$+\frac{ab}{k}K_0\left(\beta_m,\mathrm{b}\right)\frac{\sqrt{2}}{\eta_p h}(1-\cos\eta_p h)\left[1-E_\alpha(-a(\beta_m^2+\eta_p^2))t^\alpha\right]\Bigg\} \qquad (20)$$

Substituting equation (20) into equation (16), one obtains,

$$\frac{\omega(\mathrm{r},t)}{X}=\sum_{p=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{\eta_p^2}\sin\left(\eta_p h\right)\frac{1}{\beta_m^2}\frac{\sqrt{2}}{b}\left[\frac{J_0(\beta_m r)-J_0(\beta_m\,\mathrm{b})}{J_0(\beta_m,b)}\right]$$
$$\times\Bigg\{\frac{a}{k}K_0(\eta_p,z_1)K_0(\beta_m,\mathrm{r}_1)\,\mathrm{g}_{pi}\left[t^{\alpha-1}E_{\alpha,\alpha}(-a(\beta_m^2+\eta_p^2))t^\alpha\right]$$

$$+\frac{ab}{k}K_0\left(\beta_m,\mathrm{b}\right)\frac{\sqrt{2}}{\eta_p h}(1-\cos\eta_p h)\left[1-E_\alpha(-a(\beta_m^2+\eta_p^2))t^\alpha\right]\Bigg\} \tag{21}$$

4. Numerical Calculations

4.1. Dimensions

For the sake of convenience, we choose,
Radius of a thin circular plate $b=10\ m$.
Thickness of a thin circular plate $h=1m$.
Central circular paths of circular plate in radial and axial direction $r_1=5\ m$, $z_1=0.5m$, $Q_0=100$.
The heat source $g(r,z,t)$ is an instantaneous point heat source of strength $g_{pi}=500\ J/m^3$ situated at the center of the thin circular plate along the radial and axial directions releases its heat instantaneously at time $t=5$ sec.

4.2. Material Properties

The numerical calculation has been carried out for a Copper (Pure) thin circular plate with the material properties as,
Thermal diffusivity $a=112.34\times10^{-6}(m^2s^{-1})$,
Thermal conductivity $k=386(W/mk)$,
Density $\rho=8954kg/m^3$,
Specific heat $c_p=383J/kgK$,
Poisson ratio $\nu=0.35$,
Coefficient of linear thermal expansion $a_t=16.5\times10^{-6}\frac{1}{K}$,
Lamé constant $\mu=26.67$.

4.3. Roots of transcendental equation

$\beta_1=3.8317,\ \beta_2=7.0156,\ \beta_3=10.1735,\ \beta_4=13.3237,\ \beta_5=16.470$,
$\beta_6=19.6159,\ \beta_7=22.7601,\ \beta_8=25.9037,\ \beta_9=29.0468,\ \beta_{10}=32.18$
are the roots of transcendental equation $J_1(\beta b)=0$.

We set for convenience, $X = \dfrac{-a_t E h}{D(1-\nu)}$.

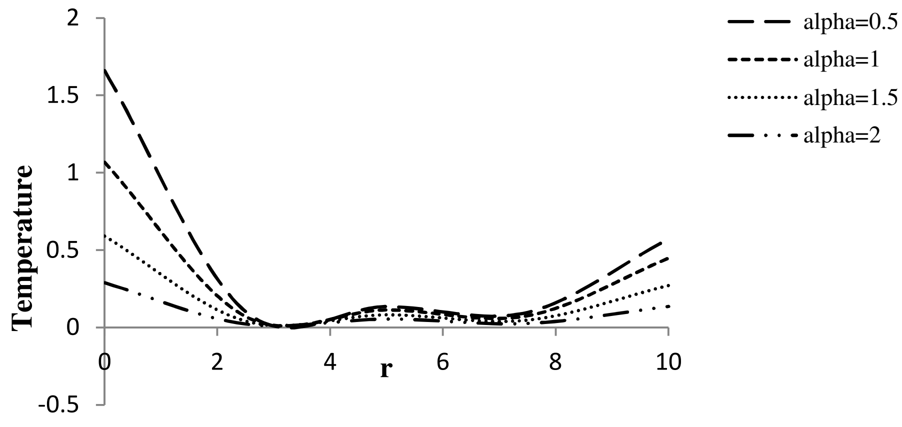

Fig.1: Temperature Distribution Function

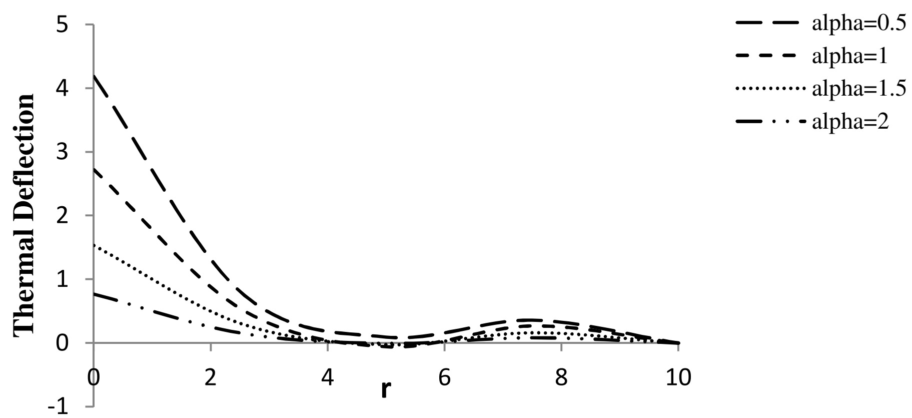

Fig.2: Thermal Deflection Function

Figures 1-2, depict the distributions of temperature and thermal deflection against the radius of the thin circular plate. The curves are plotted for different values of the fractional order parameter α.
The numerical calculation has been carried out in MATLAB 2013a programming environment.

From Figure 1, it is observed that the temperature is high at the center of the plate and along its edges, whereas the temperature decreases to zero at $r = 3$ and then follows a sinusoidal pattern.

From Figure 2, it is seen that the maximum deflection is observed towards the center of the thin circular plate and it is identically zero at the outer edge which is matching the boundary condition considered.

It is also observed that the thermal deflection decreases as we move away from the center of the plate till $r = 5$, then increases gradually till $r = 7.5$ and then becomes zero at the outer edge.

5. Conclusion

A study of thermal deflection based on time fractional order quasi static thermoelasticity with an instantaneous point heat source of strength g_{pi} situated at the center of a thin circular plate along the radial direction and axial direction, releases its heat instantaneously at the time t sec is made. The most important advantage of using fractional differential equations is its Non-local property. The time fractional differential operator describes memory effects. It is observed that when $\alpha = 0$, it predicts the localized heat conduction, $\alpha = 1$ simply diffusion equation and when $\alpha = 0.5$ its predict fractional order. Mainly it is observed that fractional order parameters predicted the interpolation of classical uncoupled thermal deflection. And also, due to the presence of the instantaneous heat source, we can observed the variation of thermal deflection for different values of the fractional order parameter α. Thus this paper may prove to be useful to the researchers in solid mechanics, designers of new materials, etc.

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