Transformations and Exact Solutions of certain time fractional Partial Differential Equatiions

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***Abstract*: Lie Symmetry analysis is used to reduce the number of independent variables of time fractional differential equations. By using symmetry properties some exact solutions are derived.**

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1. INTRODUCTION

Lie Symmetry Analysis is one of the important method to study the differential equation with integer order [1,2,3]. Through Lie Analysis construction of new solution from trivial ones, linearization of some non linear equation, deriving of integrating factor, reducing order, and reducing the number of independent variables, can be done. In recent years, fractional differential equations are widely used to describe many phenomena in various fields of science such as control theory, physics, chemistry, fluid flow[4], and so on. Adomians's decomposition method[5] the finite difference method[6], the first integral method[7], the variational iteration method[8], are powerful and efficient methods for seeking exact and numerical solutions of partial differential equations. In case of symmetry analysis of fractional partial differential equation, the Lie symmetry analysis method was started by Gazizovet at[9]. in where they have Riemann-Liouvillee and Caputo equations as basic fractional derivatives..

Now let us consider the Lie symmetry prolongation formula for time fractional equation as

$$\frac{\partial^k z}{\partial t^k} = u\frac{\partial^2 z}{\partial u^2} + f(u)\frac{\partial z}{\partial u} \qquad (1)$$

Where $\frac{\partial^k z}{\partial t^k}$ is the fractional derivative of order k with 0< k<1. The above equation has been extensively studied for k=1 by Craddock[9], with drift function f which satisfies a family of Riccati type differential equations. Now, in this paper we extend the work for some more drift functions to obtain some special exact solutions.

The paper is organised as follows: In the second section, system of symmetry equations to drive the symmetry generators are obtained. In the third section the obtained equations are solved for infinitesimal generators. In fourth section some special exact solutions of equation (1) are pointed and finally conclusion is given.

2. DERIVATION OF SYSTEM OF DETERMINING EQUATIONS VIA LIE SYMMETRY ANALYSIS:

The time fractional partial differential equation for (1) is of the form

$$\frac{\partial^k z}{\partial t^k} = F(t,u,z,z_u, z_{uu}, \ldots\ldots),\ 0<k<1$$

where subscripts denots the partial derivaties . Equation (1) is invariant under a one parameter family of continous transformations[1,2]

$$t^* = t + \varepsilon\tau(t,u,z) + O(\varepsilon),$$
$$u^* = u + \varepsilon\xi(t,u,z) + O(\varepsilon),$$
$$z^* = z + \varepsilon\varphi(t,u,z) + O(\varepsilon) \quad (2)$$

with corresponding Lie infinitesimal generator given by

$$X = \tau(t,u,z)\frac{\partial}{\partial t} + \xi(t,u,z)\frac{\partial}{\partial u} + \varphi(t,u,z)\frac{\partial}{\partial z}$$

in which $\tau\ \xi\ \varphi$ are infinitesimal to be determined.

To apply Lie Algorithem let us extend the infitesimal generator X to X^k of the form

$$X^k = X + \varphi^u\frac{\partial}{\partial z_u} + \varphi^{uu}\frac{\partial}{\partial z_{uu}} + \varphi^k\frac{\partial}{\partial z^k}$$

such that $z^k = \frac{\partial^k z}{\partial t^k}$ and $\varphi^u, \varphi^{uu}, \varphi^k$ are infinitesimals of order 1,2, and k respectively. φ^u and φ^{uu} have the form

$$\varphi^u = \varphi_u + (\varphi_z - \xi_u)z_u - \tau_u z_t - \xi_z z_u^2 - \tau_z z_u z_t \quad (3)$$

$$\varphi^{uu} = \varphi_{uu} + (2\varphi_{uz} - \xi_{uu})z_u - \tau_{uu}z_t + (\varphi_{zz} - 2\xi_{uz})z_u^2 - 2\tau_{uz}z_u z_t - \xi_{zz}z_u^3 - \tau_{zz}z_u^2 z_t + (\varphi_z - 2\xi_u)z_{uu} - 2\tau_u z_{ut} - 3\xi_z z_u z_{uu} - \tau_z z_{uu} z_t - 2\tau_z z_u z_{ut} \quad (4)$$

For the k^{th} extended infinitesimal φ^k , let us consider the Riemann-Liouville[10,11] fractional time derivative form defined as follows:

$$D^k z(t,u) = \frac{\partial^n z}{\partial t^k} = \begin{cases} \frac{\partial^n u}{\partial t^n}, & k = n \in N \\ \frac{1}{\Gamma(n-k)}\frac{\partial^n}{\partial t^n}\int_0^t \frac{z(v,u)}{(t-v)^{k+1-n}}dv, & n-1<k<n, n \in N^* \end{cases}$$

Where Γ is the well known gamma function and D^k satisfies

$$D^k t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-k)}t^{\gamma-k} \quad , k>0,\ \gamma>-1,\ t>0$$

$$D^k = \frac{t^{-k}}{\Gamma(1-k)} \quad , k \geq 0,\ t>0,$$

Hence the k^{th} extended infinitesimal φ^k has the following form

$$\varphi^k = D_t^k(\varphi) + \xi D_t^k(z_u) - D_t^k(\xi u_x) + D_t^k(D_t(\tau)z) - D_t^{k+1}(\tau_z) + \tau D_t^{k+1}(z).$$

where D_t^k denotes the total time fractional derivative.

Then this can be rewritten as[10]

$$\varphi^k = \frac{\partial^k \varphi}{\partial t^k} + (\varphi_z - k D_t(\tau))\frac{\partial^k z}{\partial t^k} - z\frac{\partial^k \varphi_z}{\partial t^k} + \delta + \sum_{n=1}^{\infty}\left[\binom{k}{n}\frac{\partial^n \varphi_z}{\partial t^n} - \binom{k}{n+1}D_t^{n+1}(\tau)\right]D_t^{k-n}(z)$$
$$-\sum_{n=1}^{\infty}\binom{k}{n}D_t^n(\xi)D_t^{k-n}(z_u)$$

Where

$$\delta = \sum_{n=2}^{\infty}\sum_{m=2}^{n}\sum_{q=2}^{m}\sum_{q=0}^{k-1}\binom{k}{n}\binom{n}{m}\binom{q}{r}\frac{1}{q!}\mathrm{x}\frac{t^{n-k}}{\Gamma(n+1-k)}[-z]^{r}\frac{\partial^{m}}{\partial t^{m}}(u^{q-r})\frac{\partial^{n-m+q}\varphi}{\partial t^{n-m}\,\partial u^{q}}$$

…… (6)

Here the item δ vanishes because $\varphi_{zz} = 0$. Eq (1) is invariant under the transformation (2) if and only if it sstisfies the condition for variance.

i.e $\varphi^{k}(z_{uu} + f'(u)z_{u})\xi$ – f(u) $\varphi^{u} - u\varphi^{uu} = 0$, whenever $z^{k} = uz_{uu} + f(u)z_{x}$ (7)

in which f is the drift function. In order to get general form of infinitesimal ξ, τ and φ substitute (3),(4) and (5) in (7) in which z_u, z_{uu}, z_t, z_{ut} …. and $D_t^{k-n} z,\ D_t^{k-n} z_u$ for n=1,2……are considered on independent variable. Hence, we have the following system af determining equation.

$$\varphi_{zz} = \xi_t - \xi_z = \tau_u = \tau_z \quad (8)$$

$$u(\varphi_z - z\,\xi_u) + \xi + ku\tau_t - u\varphi_z = 0 \quad (9)$$

$$u(z\varphi_{uz} - \xi_{uu}) - f(u)\xi_u + f^{!}(u)\xi + k\,f(u)\tau_t = 0 \quad (10)$$

$$\frac{z\partial^{k}\varphi_z}{\partial t^{k}} f(u)\varphi_u + u\varphi_{uu} - \frac{\partial^{k}\varphi}{\partial t^{k}} = 0 \quad (11)$$

$$\binom{k}{n+1} D_t^{n+1}\tau - \binom{k}{n}\frac{\partial^{n}\varphi_z}{\partial t^{n}} = 0\,, n = 0,1,2\ldots\ldots \quad (12)$$

By solving the above equations leads to infinitesimal generation of symmetries admitted by (1).

3. INFINITESIMAL GENERATORS OF THE TIME FRACTIONAL PARTIAL DIFFERENTIAL EQUATION:

From the equation (8) & (9) in the above system, we have

$$\xi(u) = aku + m\sqrt{u}, \quad \tau(t) = at + b.$$

Where a,b,m are arbitary constants. From $\varphi_{zz} = 0, \varphi$ must be linear in z. so,

$\varphi = A(t,u)z + B(t,u)$ for some function A (t,u) and B (t,u).

Eq (12) requires that

$\frac{\partial\varphi_z}{\partial t} = 0$. So A=A(u). We substitute these in equation (11) for deriving that the function A is an arbitary solution of equation (1). By substituting ξ, τ in equation (10), we have

$$A'(u) = -\frac{m}{8u\sqrt{u}} - \frac{1}{2}\left(a\,k + \frac{\sqrt{u}}{u}\,m\right)f'(u) + \frac{1}{4u\sqrt{u}}\,m\,f(u) \quad (13)$$

By differentitating equation (11) with respect to u, we have $f(u)\,A'(u) + u\,A''(u) = 0$ (14)

If we differentiate equation (13) with respect to u and substitute in equation (14) we have

$$\frac{m}{16\sqrt{u}}\left[8\mathfrak{L}f + 3 + 8u - \frac{d}{du}\mathfrak{L}f\right] - \frac{ak}{2}\,u\,\frac{d}{du}\mathfrak{L}f = 0 \quad (15)$$

Where $\mathfrak{L}f = \frac{f^2}{2} - f + x\,f'$ so we have f an a solution in Ricatti type equation $uf' - f + \frac{f^2}{2} = y(u)$

And equation (15) determines m and a.

Case I:

We consider $f(u) = \mu$ where in (1) μ is an arbitary constant. Here we have $u\,f'' + ff' = 0$.

Using equation (15) and (13) we obtain the following subcases.

If $\mu = \frac{1}{2}$, a Basis for Lie algebra of symmetries admitted by equation (1) is

$$X_1 = ku\frac{\partial}{\partial u} + t\frac{\partial}{\partial t}$$

$$X_2 = \sqrt{u}\frac{\partial}{\partial u},$$

$$X_3 = \frac{\partial}{\partial t},$$

$$X_4 = z\frac{\partial}{\partial z},$$
$$X_5 = B(t,u)\frac{\partial}{\partial z}.$$

Where B is an arbitary solution of equation (1), when $f(n) = m$. If $\mu = \frac{3}{2}$ the Lie algebra is spanned by X_1, X_3, X_4, X_5, and the infinitesimal generation X_6 is given by $X_6 = \frac{2u\frac{\partial}{\partial u} - 2\frac{\partial}{\partial z}}{2\sqrt{x}}$.

If the differential function does not belong to $\left\{\frac{1}{2}, \frac{3}{2}\right\}$ the possible symmetries are X_1, X_3, X_4,and X_5.

Case II:

We consider $f(u) = \frac{(1+3\sqrt{u})(1+\sqrt{u})^{-1}}{2}$.

In this case the differential function if satisfies the Ricath Equation

$f - uf' - \frac{1}{2}f^2 = \frac{3}{8}$ and hence we have

$$\frac{\partial^k z}{\partial t^k} = \frac{u\partial^2 z}{\partial u^2} + \frac{(1+3\sqrt{u})(1+\sqrt{u})^{-1}}{2}\frac{\partial z}{\partial u} \tag{16}$$

The symmetry admitted by the above equation spanned by $X_3, X_4,$ and X_5 and the infinitesimal generators X_6 and X_7 are given by

$$X_6 = k\,u\frac{\partial}{\partial u} + t\frac{\partial}{\partial t} + \frac{k\,(1+\sqrt{u})^{-1}}{2}z\frac{\partial}{\partial z}, \qquad X_7 = \frac{(2\sqrt{u}+2u)\frac{\partial}{\partial u} - z\frac{\partial}{\partial z}}{2(1+\sqrt{u})}.$$

4. CONSTRUCTION OF EXACT SOLUTION FROM THE SYMMETRIES

(i) Reduction when $f(u) = \frac{1}{2}$ with $X_2, +X_4$.

p = t and $I = ze^{-2\sqrt{u}}$ will be the invariants of the operator $X_2 + X_4$. The corresponding solution has the form $z(t,u) = \psi(t)e^{2\sqrt{u}}$ and the reduced equation will be

$$D_t^k(\psi(t) = \psi(t).$$

Hence, the above equation is a fractional order differential equation of order $2q^{-1}$ with $q > 2$. Then the solution of this equation is given by [12] as

$$\psi(t) = \frac{1}{2}\sum_{\alpha=0}^{q-1}\{E_t\ (\alpha\beta, 1) - (-1)^{q-\alpha-1}\mathrm{E_t}(-\mathrm{k}\beta, (-1)^{\mathrm{q}}\}$$

where E_t is the function defined by

$$E_t(\gamma, a) = \mathrm{t}^{\gamma}\sum_{\alpha=0}^{\infty}\frac{(\mathrm{at})^{\alpha}}{\Gamma(\gamma+1+\alpha)}$$

Hence the exact and original solution is given by

$$z,(t,u) = 2^{-1}e^{2\sqrt{u}}\sum_{\alpha=0}^{q-1}\left\{E_t\left(-\alpha\frac{\beta}{2},1\right) - (-1)^{\mathrm{q}-\alpha-1}E_t(-\alpha\frac{\beta}{2},(-1)^q\right\}$$

(ii)Reduction of $f(x) = \frac{3}{2}$ with X_5 is given by $X_5 = \frac{2u\frac{\partial}{\partial u} - z\frac{\partial}{\partial z}}{2\sqrt{u}}$

The invariants are $p = t$ and $I = z\sqrt{u}$ with solution $z(t,u) = \sqrt{u}\ \psi(t)$ and the reduced equation is $D_t^k(\psi(p)) = 0$. So the invariant solution becomes $z(t,u) = c\ \frac{t^{\beta-1}}{\sqrt{u}}$, where c is arbitary constant.

(iii) Reduction of $f(x) = \frac{(1+3\sqrt{x})(1+\sqrt{x})^{-1}}{2}$ with X_7

The operator has the form $X_7 = \frac{2\sqrt{u}(1+\sqrt{u})\frac{\partial}{\partial u} - z\frac{\partial}{\partial z}}{2(1+\sqrt{u})}$

The correspondings invariants are $p = t$ and $I = z(1+\sqrt{u})^{-1} = \psi(t)$. The invariant solution has the form $z(t,u) = (1+\sqrt{u})^{-1}\psi(t)$. Then the reduced equation of (16) becomes $D_t^{\beta}\psi(t) = 0$. The exact solution is given by $z(t,u) = d(1+\sqrt{u})^{-1} t^{k-1}$ where d is arbitary constant.

5. CONCLUSION

By using Lie symmetry analysis we transformed a time fractional partial differential equation into a single independent variable fractional partial equation and some exact solution are found .

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