Cone Metric Spaces and Fixed PointTheorem for GeneralizedT-ContractiveMappings Underc-Distance

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Abstract: **The purpose of this paper is to establish the generalization of T-contractive type mappings under c-distance in cone metric spaces see [2] and obtain unique common fixed point. The theorem is verified with proper example.**

Key words: Cone metric spaces, fixed point T- contractive mappings, c- distance.

I.INTRODUCTION:

In 2007, Huang and Zhang [1] introduced the concept of metric space, replacing the set of real numbers by an ordered Banach space, thereby they have defined the cone metric space. They have proved some fixed theorems of contractive mappings on complete cone metric space with the assumption of normality of a cone. The result in [1], were generalized by Rezapour, Sh. and Hamlbaraniin [2] omitting the assumption of normality on the cone. Subsequently many authors have generalized the results of [1]. And have studied fixed point theorems for normal and non-normal cone. In 2009 ,Beiranvand, A. et al. [3], introduced a new classes contraction mapping; $T-$contraction and $T-$contractive mappings and then they established and extended the Banach contraction principle.

Recently in 2011, Cho et al. [4] and Wang and Guo [5] defined concept of the c- distance in a cone metric space, which is a cone version of the w- distance of Kadaet al. [6]. In 2013, Fadail et el. [7] proved some fixed point theorems of $T-$contraction type mappings under the concept of the c-distance on complete cone metric spaces depended on another function. After that, many authors studied the existence of fixed point, common fixed point,coupled fixed point and common coupled fixedpoint problems using this distance in cone metric spaces (see [8-17]).

Our results generalize, extend and improved the results of Fadail et al. [7].Before presenting our theorems, we remember some notations, definitions and examples needed in our subsequent discussions.

II. PRELIMINARIES

Definition 2.1[1)]: Let E be a real Banach space and P a subset of E and 0 denote to the zero element in E. Then P is called a cone if and only if

(a) P is closed, non-empty and $P \neq \{0\}$,

(b) $a, b \in R, a, b \geq 0, x, y \in P \Rightarrow ax + by \in P$,

(c) if $x \in P$ and $-x \in P \Longrightarrow x = 0 \Leftrightarrow P \cap (-P) = \{0\}$.

Given a cone $P \subseteq E$, we define a partial ordering $\leq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We shall write $x \ll y$ if $y - x \in intP$ (where int$P$ denotes the interior of P). If $intP \neq \emptyset$, then cone P is solid. The cone P called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$$0 \leq x \leq y \Rightarrow \parallel x \parallel \leq K \parallel y \parallel.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

Definition: 2.2([1]): Let X be a non-empty set and E be a real Banach space equipped with the partial ordering $\leq$ with respect to the cone $P \subseteq E$. Suppose that the mapping $d: X \times X \to E$ satisfies the following conditions:

(d_1). $0 \leq d(x,y)$for all $x, y \in X$ and$(x,y) = 0$ if and only if $x = y$,

(d_2). $d(x,y) = d(y,x)$for all $x, y \in X$,

(d_3). $d(x,y) \leq d(x,z) + d(z,y)$for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space. The concept of cone metric space is more general than that of a metric space.

Example2.3: Let $E = R^2, P = \{(x,y) \in E: x, y \geq 0\}$, $X = R$ and $d: X \times X \to E$ defined by $d(x,y) = (|x - y|,\ \alpha|x - y|)$,where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition2.4: Let $(X,\ d)$ be a cone metric space, $x \in X$ and $\{x_n\}_{n\geq 1}$ be a sequence in X. Then,

1. $\{x_n\}_{n\geq 1}$ converges to $x$if for every $c \in E$ with $0 \ll c$there exists a nature number $n_0 \in N$ such that $d(x_n, x) \ll c$ for all $n \geq n_0$. We denote this by $lim_{n\to\infty} d(x_n, x) = 0$.

2. $\{x_n\}_{n\geq 1}$ is said to be a Cauchy sequence if for every $c \in E$ with $0 \ll c$, there existsnature number $n_0 \in N$ such that $d(x_n, x_m) \ll c$ for all $n, m \geq n_0$.

3. (X, d) is called a complete cone metric space if every Cauchy sequence in X is convergent.

Next, we give the notion of c-distance on a cone metric space (X, d) of **Cho, et al. [4]&Wang and Guo in [5],** which is a generalization of w -distance of Kadaet al. [18].

Definition 2.5 [4,5]: Let (X, d) be a cone metric space. Then a function $q: X \times X \to E$ is called a c- distance on X if the following conditions hold:

(q_1). $0 \leq q(x,y)$ for all $x, y \in X$;

(q_2). $q(x,z) \leq q(x,y) + q(y,z)$ for all $x, y, z \in X$;

(q_3). for each $x \in X$ and $n \geq 1$, if $q(x, y_n) \leq u$ for some $u = u_x \in P$, then $q(x,y) \leq u$whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$;

(q_4). for all $c \in E$ with $0 \ll c$, there exist $e \in E$ with $0 \in e$ such that $q(z,x) \ll e$ and $q(z,y) \ll e$ imply $d(x,y) \ll c$

Example 2.5: Let $E = R$ and $P = \{x \in E: x \geq 0\}$, let $X = [0,1)$ and define a mapping $d: X \times X \to E$ by $d(x, y) = |x - y|$ for all $x, y \in X$. Then (X, d) is a cone metric space. Define a mapping $q: X \times X \to E$ by $q(x, y) = y$, for all $x, y \in X$. Then q is a c-distance.

The following lemmas are very important to prove our results:

Lemma 2.6 ([4, 5,19]): Let (X, d) be a cone metric space and q be a c- distance on X. Also, let$\{x_n\}$ and $\{y_n\}$ be sequence in X and $x,\ y, z \in X$. Suppose that $\{u_n\}$ and $\{v_n\}$are two sequences in P converging to 0. Then the following conditions hold:

$(\boldsymbol{qp_1})$. If $q(x_n,\ y) \leq u_n$ and $q(x_n,\ z) \leq v_n$, for $n \in N$, then $y = z$. Specifically, if $q(x, y) = 0$ and $q(x, z)$ =0, Then $y = z$.

$(\boldsymbol{qp_2})$. If $q(x_n,\ y_n) \leq u_n$ and $q(x_n,\ z) \leq v_n$, for $n \in N$, then $\{y_n\}$ converges to z.

$(\boldsymbol{qp_3})$. If $q(x_n, x_m) \leq u_n$ for $m > n$ and $\{x_n\}$ is a Cauchy sequence in X.

$(\boldsymbol{qp_4})$. If $q(y, x_n) \leq u_n$ then $\{x_n\}$ is a Cauchy sequence in X.

Lemma 2.7 ([21]):

1. Let E be a real Banach space with a cone P and $q \leq \lambda a$ where $a \in P$ and $0 \leq \lambda < 1$, then $a =$ 0.
2. If $C \in intP$, $0 \leq a_n$ and $a_n \to 0$, then there exists a positive integer N such that $a_n \leq C$ for all $n \geq N$.

Remark 2.8 [4]

1. For c-distance $q, q(x, y) = 0$ is not necessarily equivalent to $x = y$, for all $x, y \in X$.
2. $q(x, y) = q(y, x)$ does not necessarily for all $x, y \in X$.

Definition 2.9 Let (X, d) be a cone metric space. If for any sequence $\{x_n\}$ in X, there is a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ is convergent in X. Then X is called a aequently compact cone metric space.

Definition 2.10 ([19]): Let(X, d) be a cone metric space, P be a solid cone and $T: X \to X$, then

(a) T is said to be continuous if $\lim_{n\to\infty} x_n = x$ implies that $\lim_{n\to\infty} Tx_n = Tx$ for all $\{x_n\}$ in X;

(b) T is said to be sub sequentially convergent, if for every sequence $\{x_n\}$ that $\{Tx_n\}$ is convergent, implies $\{x_n\}$ has a convergent
subsequence,

(c) T is said to be sequentially convergent if for every sequence $\{x_n\}$, if $\{Tx_n\}$ is convergent, then $\{x_n\}$ is also convergent.

Definition 2.11 (see [3]) Let (X, d) be a cone metric space and $T, S: X \to X$ two functions. A mapping $S$is said to be a T-contraction if there exists$k \in [0,1)$ such that
$d(TSx, TSy) \leq kd(Tx, Ty)$, for all $x, y \in X$.

III. MAIN RESULTS

The following theorem is extends and improves Theorem 1 from [7].

Theorem 3.1. Let (X, d) be a complete cone metric space, P be a solid cone and q be a c- distance on X.In addition, let $T: X \to X$ be a one to one and continuous function,subsequentially convergent and $R, S: X \to X$ be a pair of$T$-contraction mapping satisfies the condition

$q(TRx, TSy) \leq kd(Tx, Ty)$……………………… (3.1.1)

For all $x, y \in X$ where $k \in [0,1)$ is a constant. Then R and S have an unique common fixed point $x^* \in X$ and for any $x \in X$, iterative sequence$\{Rx_{2n}\}$ and $\{Sx_{2n+1}\}$converges to the common fixed point.If $v = Rv = Sv$. Then $q(v, v) = 0$.

Proof: Let x_0 be an arbitrary point in X. We define the iterative sequence$\{x_{2n}\}$ and $\{x_{2n+1}\}$by

$x_{2n+1} = Rx_{2n} = R^{2n}x_0$…………………………(3.1.2) and

$x_{2n+2} = Sx_{2n+1} = S^{2n+1}x_0$…………………… (3.1.3).

Then from (2.1.1), we have

$q(Tx_{2n,}Tx_{2n+1}) = q(TRx_{2n-1}, TSx_{2n})$

$\leq kq(Tx_{2n}, Tx_{2n+1})$

$\leq k^2 q(Tx_{2n-1}, Tx_{2n})$

$\leq k^3 q(Tx_{2n-2}, Tx_{2n-1})$

$\leq \dots \dots \dots$

$\leq k^{2n} q(Tx_0, T\,x_1)$………………………….. (3.1.4)

So, for $m, n \in N, m > n$, we have

$q(Tx_{2n,}Tx_{2m}) \leq q(Tx_{2n,}Tx_{2n+1}) + q(Tx_{2n+1}, Tx_{2n+2}) + \dots\dots + q(Tx_{2m-1,}Tx_{2m})$

$\leq (k^{2n} + k^{2n+1} + k^{2n+2} + \cdots . + k^{2m-1})q(Tx_0, Tx_1)$

$\leq \frac{k^{2n}}{1-k} q(Tx_0, Tx_1)$ ……………………….. (3.1.5)

Thus, lemma2.6(qp_3), which implies that,$\{TRx_{2n}\}$ is a Cauchy sequence in X. Since X is a complete cone metric space, then there exists$u \in X$ such that

$Tx_{2n} \to u$ as $n \to \infty$. …………………. (3.1.6)

Since T is subsequently convergent, $\{x_{2n}\}$ has a convergent subsequence. So, there exists $x^* \in X$ and $\{x_{2n_i}\}$ such that

$x_{2n_i} \to x^*$ as $n \to \infty$. ……………………. (3.1.7)

Since T is continuous, then from (3.1.7), we have

$Tx_{2n_i} = Tx^*$……………………… (3.1.8)

The uniqueness of the limit, from equality (2.1.6) we conclude that

$$Tx^* = u.$$

Then by 2.6(qp_3), we have

$q(Tx_{2n,}Tx^*) \leq \frac{k^{2n}}{1-k} q(Tx_0, Tx_1)$ ………………. (3.1.9)

On the other hand,and by using(3.1.3), we have

$q(Tx_{2n}, TRx^*) = q(TRx_{2n-1}, TRx^*)$

$\leq kq(Tx_{2n-1}, Tx^*)$

$\leq \ k \frac{k^{2n-1}}{1-k} q(Tx_0, Tx_1)$

$\leq \frac{k^{2n}}{1-k} q(Tx_0, Tx_1)$ ……………… (3.1.10)

By lemma1.6(qp_1), from (3.1.9) and (3.1.10), we have

$Tx^* = TRx^*$ …………………… . (3.1.11)

Since T is one to one. Then $x^* = Rx^*$.Thus x^* is a fixed point of R. Similarly, we can prove thatx^* is a fixed point of S. Therefore, x^* is a fixed point of R and S and $v = Rv = Sv$. Then we have

$$q(Tv, Tv) = q(TRv, TSv)$$

$$\leq k(Tv, Tv)$$

Since $k < 1$, then by lemma 2.7(1), shows that $q(Tv, Tv) = 0$.

Finally suppose that, if y^* is another common fixed point of $R$and S. Then we have

$$q(Tx^*, Ty^*) = q(TRx^*, TSy^*)$$

$\leq kq(Tx^*, Ty^*)$

Since $k < 1$, then by lemma 2.7(1), shows that $q(Tx^*, Ty^*) = 0$.Also we have $q(Tx^*, Tx) = 0$.Thus, 1.6(qp_1), $Tx^* = Ty^*$. Since Tis one to one, then $x^* = y^*$. So, x^* is the unique common fixed point of R and S.

This completes the proof of theorem.

Example: Let $E = (C_{[0,1]}\ R), P = \{\varphi \in E: \varphi \geq 0\}$and $X = [0, 1]$. Define $d: X \times X \to E$ by $d(x, y) = |x - y|e^t$ where $e^t \in E$.Then $(X, d)$is a complete metric space. Define a mapping

$q: X \times X \to E$by $q(x, y) = xe^t$for all $x, y \in X$. Then q is a c-distance on X.Define the mapping $T, R, S: X \to X$by $T(x) = x^3, R(x) = \frac{x}{2}$and $S(x) = \frac{x}{4}$ for all $x \in X$. Now we have

$$q(TRx, TSy) = TRS.xe^t$$

$$= TRS(x)e^t$$

$$= TR(\frac{x}{4})e^t$$

$$= T(\frac{x}{8})e^t$$

$$= \frac{x^3e^t}{512}$$

$$\leq \frac{1}{8}x^3e^t$$

$$= kq(Tx, Ty)$$

Where $k = \frac{1}{8} \in [0,1)$, therefore, all conditions of theorem 3.1 is satisfied. Hence R and S have unique common fixed point $x = 0$, $R(0) = 0$ and $S(0) = 0$.

IV. CONCLUSION

In this attempt we prove unique common fixed point theorem for generalized T-Contractive mappings in Cone metric spaces under c-distance. Our result extends and generalize the results of Fadail et al. [7].

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