On P* and of P- Reducible Cartan's second curvature tensor

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Abstract. We will develop the generalized $\mathcal{B}P$ −recurrent space which characterize by the following condition [1] $\mathcal{B}_l P^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l(\delta^i_j g_{kh} - \delta^i_k g_{jh})$, $P^i_{jkh} \neq 0$,

By using the properties for some spaces.

The purpose of the present paper to use properties of P^*- space and $P-$ reducible space in the generalized $\mathcal{B}P$ −recurrent space and we get new spaces which we will call P^*- generalized $\mathcal{B}P$ −recurrent space and $P-$ reducible generalized $\mathcal{B}P$ −recurrent space, respectively. Also we obtain different theorems for some tensors which satisfy in above spaces.

Keywords: *Finsler space, P^*- generalized $\mathcal{B}P-$ recurrent space, $P-$ reducible generalized $\mathcal{B}P-$ recurrent space.*

1. Introduction

Izumi ([10],[11]) gave the concept of P^*- Finsler space which was a generalized of C^h–recurrent space, Mohammed [2] introduced P^* −Finsler spaces in P^h- recurrent space, Awed [3] studied P^* −Finsler spaces in generalized P^h- recurrent space, Qasem and Abdallah [5] discussed P^* −Finsler spaces in a generalized $\mathcal{B}R$ – recurrent space, Qasem and Baleedi [8] introduced P^* − Finsler spaces in a generalized $\mathcal{B}K$ −recurrent space, Qasem and Al– Qashbari [7] studied P^*- spaces in a generalized H^h- recurrent Finsler space, Al – Qashbari [4] introduced P^*- spaces in a generalized R^h- recurrent Finsler space.

Dwivedi[14] introduced a $P-$ reducible Finsler spaces and Applications, Verma [17] obtained the condition for a $P-$ reducible R^h- recurrent space to be necessarily a Landsberg space, Swaroop [16] had worked out the role of $P-$ reducibility condition in spacial Finsler spaces, Qasem and Abdallah [6] studied $P-$ reducible in a generalized $\mathcal{B}R$ – recurrent space, Qasem and Baleedi [9] introduced $P-$ reducible in a generalized $\mathcal{B}K$ −recurrent space, Awed [3] introduced

$P-$ reducible in generalized P^h- Recurrent Space, Pandy and Dikshit [15] studied P^*- and $P-$ ruducible Finsler spaces of recurrent curvature.

Let F_n be a n – dimensional Finsler space equipped with the metric function *F*(*x*,*y*) satisfying the request conditions [12]

The vector y_i is defined by

(1.1) $y_i = g_{ij}(x,y)y^j$

The two sets of quantities g_{ij} and its associative g^{ij}, which are components of a metric tensor connected by

(1.2) $g_{ij}g^{ik} = \delta_j^k = \begin{cases} 1 & if \ j = k\,, \\ 0 & if \ j \neq k\,. \end{cases}$

In view of (1.1) and (1.2), we have

(1.3) a) $\delta_k^i y^k = y^i$, b) $\delta_j^i g_{ir} = g_{jr}$ and c) $\delta_k^i y_i = y_k$

The tensor C_{ijk} is defined by

$$C_{ijk} = \frac{1}{2}\dot{\partial}_k g_{ij}$$

which called $(h)hv$ *–torsion tensor* [13] and its associative $C^{\,i}_{jk}$ is symmetric in its lower indices and called $(v)hv-$ *torsion tensor*, these tensors satisfy the following:

(1.4) a) $C^i_{jk}\, y_i = 0$, b) $C_{ijk}g^{hj} = C^h_{ik}$

c) $C^i_{ri} = C_r$, d) $g_{hj}C^h_{ik} = C_{ijk}$

e) $\delta^i_r C^r_{jh} = C^i_{jh}$ and f) $\delta^r_j C_{irk} = C_{ijk}$

Berwald covariant derivative $\mathcal{B}_k T^i_j$ of an arbitrary tensor field T^i_j with respect to x^k is given by [12]

$$\mathcal{B}_k T^i_j := \partial_k T^i_j - (\dot{\partial}_r T^i_j)G^r_k + T^r_j G^i_{rk} - T^i_r G^r_{jk}.$$

Berwald covariant derivative of the vector y^i vanish identically, i.e.

(1.5) $\mathcal{B}_k y^i = 0$

But, in general, Berwald covariant derivative of the metric tensor g_{ij} does not vanish and given by

(1.6) $\mathcal{B}_k g_{ij} = -2y^h \mathcal{B}_h C_{ijk}$

The hv –curvature tensor P^i_{jkh} (Cartan's second curvature tensor) is defined by [12]

(1.7) $P^i_{jkh} := \dot{\partial}_h \Gamma^{*i}_{jk} + C^i_{jr}P^r_{kh} - C^i_{jh|k}$.

The tensor P^i_{jkh} is positively homogeneous of degree zero in y^i and satisfies

(1.8) $P^i_{jkh} y^j = P^i_{kh}$

where P^i_{jk} is called the $v(hv)$ −torsion tensor and its associative tensor P_{rkh} is given by

(1.9) $a)\ g_{ir}P^i_{kh} = P_{rkh}$ and b) $P_{rkh}\, g^{ir} = P^i_{kh}$

The associate curvature tensor P_{rjkh} is given by

(1.10) $a)\ P^i_{jkh}g_{ir} = P_{rjkh}$ and b) $P_{ijkh}g^{ir} = P^r_{jkh}$.

The curvature vector P_k and the curvature scalar P of Cartan's second curvature tensor are given by

(1.11) $P^i_{ki} = P_k$

and

(1.12) $P_k\, y^k = P$

The tensor S^i_{jkh} is called $v -$ *curvature tensor* (*Cartan's first curvature tensor*) defined by [12]

(1.13) $S^i_{jkh} = C^i_{rk}C^r_{jh} - C^i_{rh}C^r_{jk}$.

The associate curvature tensor S_{ijkh} is given by

(1.14) $S_{ijkh} = g_{ri}\, S^r_{jkh}$

In contracting the indices i and h in (1.13), we get

(1.15) $S^i_{jki} = S_{jk} = C^s_{rk}C^r_{js} - C_r C^r_{jk}$

The deviation tensor S^i_j is given by

(1.16) $S^i_j = g^{ik}S_{kj}$

The scalar S is given by

(1.17) $S = g^{jh}S_{jh}$.

2. A Generalized $\mathcal{B}P-$ Recurrent Space

Let us consider a Finsler space F_n which Cartan's second curvature tensor P^i_{jkh} satisfies the condition [1]

(2.1) $\mathcal{B}_l P^i_{jkh} = \lambda_l P^i_{jkh} + \mu_l\left(\delta^i_j g_{kh} - \delta^i_k g_{jh}\right)$, $P^i_{jkh} \neq 0$

where $\mathcal{B}_l$ is Berwald's covariant differential operator with respect to x^l, λ_l and μ_l called *recurrence vectors*. This space introduced by Alaa et. al. [1], they called *it a generalized* $\mathcal{B}P$ − *recurrent space and briefly denoted by* $G(\mathcal{B}P) - RF_n$.

Let us consider a $G(\mathcal{B}P) - RF_n$ which is characterized by the condition (2.1). Transvecting the condition (2.1) by g_{im} , using (1.10a), (1.6) and (1.3b), we get

(2.2) $\mathcal{B}_l P_{mjkh} = \lambda_l P_{mjkh} + \mu_l (g_{jm} g_{kh} - g_{km} g_{jh}) + 2P^i_{jkh} y^t \mathcal{B}_t C_{iml}$.

Transvecting the condition (2.1) by y^j, using (1.8), (1.5), (1.3a) and (1.1), we get

(2.3) $\mathcal{B}_l P^i_{kh} = \lambda_l P^i_{kh} + \mu_l (y^i g_{kh} - \delta^i_k y_h)$.

Transvecting the condition (2.3) by g_{ij} , using (1.9a), (1.6), (1.1) and (1.3b), we get

(2.4) $\mathcal{B}_l P_{jkh} = \lambda_l P_{jkh} + \mu_l (g_{kh} y_j - g_{kj} y_h) + 2P^i_{kh} y^t \mathcal{B}_t C_{ijl}$.

Contracting the indices i and h in the condition (2.3), using (1.11), (1.1) and (1.3c), we get

(2.5) $\mathcal{B}_l P_k = \lambda_l P_k$.

Transvecting the condition (2.5) by y^K, using (1.12) and (1.5), we get

(2.6) $\mathcal{B}_l P = \lambda_l P$.

3. P^* − Generalized $\mathcal{B}P$ − Recurrent Space

A P^* − Finsler space is characterize by the condition [9]

(3.1) $P^i_{kh} = \varphi C^i_{kh}$,

Where

$$P^i_{jkh} y^j = P^i_{kh} = C^i_{kh|s} y^s .$$

Definition 3.1. *The generalized $\mathcal{B}P$ − recurrent space which satisfies the property of P^* − space [characterized by the condition (3.1)], we will call a P^* − generalized $\mathcal{B}P$ − recurrent space and will denote it briefly by $P^* - G(\mathcal{B}R) - RF_n$.*

Let us consider a $P^* - G(\mathcal{B}R) - RF_n$.

Taking the covariant derivative for the condition (3.1) with respect to x^l in the sense of Berwald and using (2.3) we get

(3.2) $\mathcal{B}_l (\varphi C^i_{kh}) = \lambda_l P^i_{kh} + \mu_l (y^i g_{kh} - \delta^i_k y_h)$

Using (3.1) in (3.2), we get

(3.3) $\mathcal{B}_l (\varphi C^i_{kh}) = \lambda_l (\varphi C^i_{kh}) + \mu_l (y^i g_{kh} - \delta^i_k y_h)$

Thus, we conclude

Theorem 3.1. *In $P^* - G(\mathcal{B}P) - RF_n$, the tensor (φC^i_{kh}) is given by Eq. (3.3).*

Transvecting (3.1) by g_{ij} , using (1.9a) and (1.4d), we get

(3.4) $P_{jkh} = \varphi C_{jkh}$

Taking the covariant derivative for (3.4) with respect to x^l in the sense of Berwald and using (2.4), we get

(3.5) $\mathcal{B}_l (\varphi C_{jkh}) = \lambda_l P_{jkh} + \mu_l (g_{kh} y_j - g_{kj} y_h) + 2P^i_{kh} y^t \mathcal{B}_t C_{iml}$

Using (3.4) in (3.5), we get

(3.6) $\mathcal{B}_l(\varphi C_{jkh}) = \lambda_l(\varphi C_{jkh}) + \mu_l(g_{kh}y_j - g_{kj}y_h) + 2P^i_{kh}y^t\mathcal{B}_tC_{iml}$

This shows that

(3.7) $\mathcal{B}_l(\varphi C_{jkh}) = \lambda_l(\varphi C_{jkh}) + \mu_l(g_{kh}y_j - g_{kj}y_h)$

if and only if

$$y^t\mathcal{B}_tC_{iml} = 0$$

Since $P^i_{kh} \neq 0$

Thus, we conclude

Theorem 3.2. *In $P^* - G(\mathcal{B}P) - RF_n$, the tensor (φC_{jkh}) is given by Eq. (3.7) if and only if $y^t\mathcal{B}_tC_{iml} = 0$*

Contracting the indices i and h in (3.1), using (1.11) and (1.4c), we get

(3.8) $P_k = \varphi C_k$

Taking the covariant derivative for (3.8) with respect to x^l in the sense of Berwald, and using (2.5), we get

(3.9) $\mathcal{B}_l(\varphi C_k) = \lambda_l P_k$

Using (3.8) in (3.9), we get

(3.10) $\mathcal{B}_l(\varphi C_k) = \lambda_l(\varphi C_k)$

Transvecting (3.8) by y^k, using (1.12) and put ($C_k y^k = C$), we get

(3.11) $P = \varphi C$

Taking the covariant derivative for (3.11) with respect to x^l in the sense of Berwald and using (2.6), we get

(3.12) $\mathcal{B}_l(\varphi C) = \lambda_l P$

Using (3.11) in (3.12), we get

(3.13) $\mathcal{B}_l(\varphi C) = \lambda_l(\varphi C)$

The equations (3.10) and (3.13) show that the tensors (φC_k) and (φC) behave as recurrent.

Thus, we conclude

Theorem 3.3. *In $P^* - G(\mathcal{B}P) - RF_n$, the tensors (φC_k) and (φC) behave as recurrent.*

Suppose that the $v(hv)-$ torsion tensor C^i_{kh} and the $h(hv)-$ torsion tensor C_{jkh} satisfy

(3.14) $a)$ $\mathcal{B}_l C^i_{kh} = \lambda_l C^i_{kh} + \mu_l(\delta^i_k y_h - \delta^i_h y_k)$

and

$$b)\ \mathcal{B}_l C_{jkh} = \lambda_l C_{jkh} + \mu_l \left(g_{jk} y_h - g_{jh} y_k \right),$$

respectively.

Taking the covariant derivative for (1.13) with respect to x^l in the sense of Berwald, we get

(3.15) $\mathcal{B}_l\, S^i_{jkh} = (\mathcal{B}_l C^i_{rk}) C^r_{jh} + C^i_{rk}(\mathcal{B}_l C^r_{jh}) - (\mathcal{B}_l C^i_{rh}) C^r_{jk} - C^i_{rh}(\mathcal{B}_l C^r_{jk})$.

Using (3.14a) in (3.15), we get

(3.16) $\mathcal{B}_l\, S^i_{jkh} = 2\lambda_l (C^i_{rk} C^r_{jh} - C^i_{rh} C^r_{jk}) + \mu_l (\delta^i_r y_k C^r_{jh} - \delta^i_k y_r C^r_{jh} + \delta^r_j y_h C^i_{rk}$
$-\delta^r_h y_j C^i_{rk} - \delta^i_r y_h C^r_{jk} + \delta^i_h y_r C^r_{jk} - \delta^r_j y_k C^i_{rh} + \delta^r_k y_j C^i_{rh})$

Using (1.13), (1.4a) and (1.4e) in (3.16), we get

(3.17) $\mathcal{B}_l\, S^i_{jkh} = \alpha_l\, S^i_{jkh}$.

Where $\alpha_l = 2\lambda_l$

Transvecting (1.13) by g_{im} , using (1.14) and (1.4d), we get

(3.18) $S_{mjkh} = C_{mrk} C^r_{jh} - C_{mrh} C^r_{jk}$.

Taking the covariant derivative for (3.18) with respect to x^l in the sense of Berwald, we get

(3.19) $\mathcal{B}_l\, S_{mjkh} = (\mathcal{B}_l C_{mrk}) C^r_{jh} + C_{mrk}(\mathcal{B}_l C^r_{jh}) - (\mathcal{B}_l C_{mrh}) C^r_{jk} - C_{mrh}(\mathcal{B}_l C^r_{jk})$.

Using (3.14a) and (3.14b) in (3.19), we get

(3.20) $\mathcal{B}_l\, S_{mjkh} = 2\lambda_l (C_{mrk} C^r_{jh} - C_{mrh} C^r_{jk}) + \mu_l (g_{mr} y_k C^r_{jh} - g_{mk} y_r C^r_{jh}$
$+\delta^r_j y_h C_{mrk} - \delta^r_h y_j C_{mrk} - g_{mr} y_h C^r_{jk} + g_{mh} y_r C^r_{jk}$
$-\delta^r_j y_k C_{mrh} + \delta^r_k y_j C_{mrh})$.

Using (3.18), (1.4a), (1.4d) and (1.4f) in (3.20), we get

(3.21) $\mathcal{B}_l\, S_{mjkh} = \alpha_l\, S_{mjkh}$

Where $\alpha_l = 2\lambda_l$

Contracting the indices i and h in (3.17), using (1.15), we get

(3.22) $\mathcal{B}_l\, S_{jk} = w_l\, S_{jk}$

The equations (3.17), (3.21) and (3.22) show that the tensors S^i_{jkh} , S_{mjkh} and S_{jk} behave as recurrent.

Thus, we conclude

Theorem 3.4. *In $P^* - G(\mathcal{B}P) - RF_n$, Cartan first curvature tensor S^i_{jkh} , associative curvature tensor S_{mjkh} and $S-$ Ricci tensor S_{jk} behave as recurrent.*

Transvecting (3.22) by g^{ij}, using (1.16), we get

$$\mathcal{B}_l\, S_k^i = \alpha_l\, S_k^i + S_{jk}\mathcal{B}_l g^{ij}$$

This shows that

(3.23) $\mathcal{B}_l\, S_k^i = w_l\, S_k^i$

if and only if

$$\mathcal{B}_l g^{ij} = 0$$

Since $S_{jk} \neq 0$

Transvecting (3.22) by g^{jk}, using (1.17), we get

$$\mathcal{B}_l S = w_l S + S_{jk}\mathcal{B}_l g^{jk}$$

This shows that

(3.24) $\mathcal{B}_l\, S = w_l\, S$

if and only if

$$\mathcal{B}_l g^{jk} = 0$$

Since $S_{jk} \neq 0$

The equations (3.23) and (3.24) show that the division tensor S_k^i and curvature scalar S behave as recurrent if and only if $\mathcal{B}_l g^{ij} = 0$ and $\mathcal{B}_l g^{jk} = 0$, respectively.

Thus, we conclude

Theorem 3.5. *In $P^* - G(\mathcal{B}P) - RF_n$, the division tensor S_k^i and curvature scalar S behave as recurrent if and only if $\mathcal{B}_l g^{ij} = 0$ and $\mathcal{B}_l g^{jk} = 0$, respectively.*

4. P − Reducible − Generalized $\mathcal{B}P$ − Recurrent Space

A $P -$ reducible space is characterized by any condition ([14], [15], [16])

(4.1) $P_{jkh} = \lambda C_{jkh} + \varphi(h_{jk}C_h + h_{jh}C_k + h_{kh}C_j)$,

Where λ and φ are scalar vectors positively homogeneous of degree one in y^j and h_{jk} is the angular metric tensor.

(4.2) $P_{jkh} = \frac{1}{n-1}(P_j h_{kh} + P_k h_{hj} + P_h h_{jk})$,

where $P_{jkh} = C_{jkh|m} y^m$, $P_{ik}^i = P_k$ and $h_{ij} \coloneqq g_{ij} - l_i l_j$.

Definition 4.1. *The generalized $\mathcal{B}P -$ recurrent space* which satisfies the property *of $P -$ reducible space [characterized by conditions (4.1) or (4.2)], we will call a $P -$ reducible generalized $\mathcal{B}P -$ recurrent space and will denote it briefly by $P - reducible - G(\mathcal{B}P) - RF_n$.*

In $P -$ reducible space, the hv −curvature tensor P_{ijkh} is given by [14]

(4.3) $P_{ijkh} = \left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}$

Where a) $\emptyset_j = \lambda_j - \varphi C_j$,

b) $E_{kj} = C_k \varphi_j + \varphi \dot{\partial}_j C_k + \varphi F^{-1}\left(L_j C_k + L_k C_j\right)$,

c) $B_{hj} = C_h \varphi_j + \varphi C_{h|j} + \varphi F^{-1}\left(L_h C_j + L_j C_h\right)$,

d) $\lambda_j = \dot{\partial}_j \lambda$,

e) $\varphi_j = \dot{\partial}_j \varphi$

and f) $F^{-1} = 1/F$, F is the fundamental function of Finsler space.

Let us consider a $P -$ reducible $- G(\mathcal{B}P) - RF_n$

Taking the covariant derivative for the condition (4.3) with respect to x^l in the sense of Berwald and using (2.2), we get

(4.4) $\mathcal{B}_l \left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}\right\} = \lambda_l P_{ijkh}$

$+\mu_l\left(g_{ij} g_{kh} - g_{ik} g_{jh}\right) + 2P^i_{jkh} y^t \mathcal{B}_t C_{rml}$.

Using (4.3) in (4.4), we get

(4.5) $\mathcal{B}_l \left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}\right\}$

$= \lambda_l \left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}\right\}$

$+ \mu_l\left(g_{ij} g_{kh} - g_{ik} g_{jh}\right) + 2P^i_{jkh} y^t \mathcal{B}_t C_{rml}$.

This shows that

(4.6) $\mathcal{B}_l \left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}\right\}$

$= \lambda_l \left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}\right\}$

$+ \mu_l\left(g_{ij} g_{kh} - g_{ik} g_{jh}\right)$

if and only if

$y^t \mathcal{B}_t C_{rml} = 0$.

Since $P^i_{jkh} \neq 0$

Thus, we conclude

Theorem 4.1. *In* $P - reducible - G(\mathcal{B}P) - RF_n$ *, the tensor* $\left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^{i}/_{j}\right) - \lambda\, S_{ijkh}\right\}$ *is a generalized recurrent if and only if* $y^t \mathcal{B}_t C_{rml} = 0$.

Transvecting (4.3) by g^{ir}, using (1.10b) , we get

(4.7) $P^r_{jkh} = g^{ir}\left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^i/_j\right) - \lambda\, S_{ijkh}\right\}$

Taking the covariant derivative for (4.7) with respect to x^l in the sense of Berwald and using (2.1), we get

(4.8) $\mathcal{B}_l\left[g^{ir}\left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^i/_j\right) - \lambda\, S_{ijkh}\right\}\right]$

$$= \lambda_l P^r_{jkh} + \mu_l\left(\delta^r_j g_{kh} - \delta^r_k g_{jh}\right).$$

Using (4.7) in (4.8), we get

(4.9) $\mathcal{B}_l\left[g^{ir}\left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^i/_j\right) - \lambda\, S_{ijkh}\right\}\right]$

$$= \lambda_l\left[g^{ir}\left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^i/_j\right) - \lambda\, S_{ijkh}\right\}\right]$$

$$+ \mu_l\left(\delta^r_j g_{kh} - \delta^r_k g_{jh}\right).$$

Thus, we conclude

Theorem 4.2. *In* $P-reducible-G(\mathcal{B}P)-RF_n$, *the tensor* $g^{ir}\left\{\left(\emptyset_j C_{ikh} + \varphi_j h_{kh} C_i + E_{kj} h_{ih} + B_{hj} h_{ik} - {}^i/_j\right) - \lambda\, S_{ijkh}\right\}$ *is a generalized recurrent.*

Taking the covariant derivative for the condition (4.1) with respect to x^l in the sense of Berwald and using (2.4), we get

(4.10) $\mathcal{B}_l\{\lambda C_{jkh} + \varphi(h_{jk}C_h + h_{jh}C_k + h_{kh}C_j)\} = \lambda_l P_{jkh} + \mu_l\left(g_{kh}y_j - g_{kj}y_h\right)$

$$+2P^i_{kh}y^t\mathcal{B}_t C_{ijl}.$$

Using (4.1) in (4.10), we get

(4.11) $\mathcal{B}_l\{\lambda C_{jkh} + \varphi(h_{jk}C_h + h_{jh}C_k + h_{kh}C_j)\} = \lambda_l\{\lambda C_{jkh} + \varphi(h_{jk}C_h + h_{jh}C_k$

$$+ h_{kh}C_j)\} + \mu_l\left(g_{kh}y_j - g_{kj}y_h\right) + 2P^i_{kh}y^t\mathcal{B}_t C_{ijl}.$$

Transvecting (4.1) by g^{ij}, using (1.9b) and (1.4b), we get

(4.12) $P^i_{kh} = \lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh}C^i)$,

Where $h^i_k = g^{ij}h_{jk}$ and $C^i = g^{ij}C_j$

Taking the covariant derivative for (4.12) with respect to x^l in the sense of Berwald and using (2.3), we get

(4.13) $\mathcal{B}_l\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh}C^i)\} = \lambda_l P^i_{kh} + \mu_l\left(y^i g_{kh} - \delta^i_k y_h\right)$

Using (4.12) in (4.13), we get

(4.14) $\mathcal{B}_l\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh}C^i)\} = \lambda_l\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k$

$$+ h_{kh}C^i)\} + \mu_l\left(y^i g_{kh} - \delta^i_k y_h\right).$$

show that the tensors $\{\lambda C_{jkh} + \varphi(h_{jk}C_h + h_{jh}C_k + h_{kh}C_j)\}$ and $\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh}C^i)\}$ are given by (4.11) and (4.14), respectively.

Thus, we conclude

Theorem 4.3. *In $P - reducible - G(\mathcal{B}P) - RF_n$, the tensors $\{\lambda C_{jkh} + \varphi(h_{jk}C_h + h_{jh}C_k + h_{kh}C_j)\}$ and $\{\lambda\, C^i_{kh} + \varphi(h^i_k C_h + h^i_h C_k + h_{kh}C^i)\}$ are given by Eqs. (4.11)-(4.14), respectively.*

Taking the covariant derivative for the condition (4.2) with respect to x^l in the sense of Berwald and using (2.4), we get

(4.15) $$\mathcal{B}_l\left\{\frac{1}{n-1}\left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right)\right\} = \lambda_l P_{jkh} + \mu_l\left(g_{kh}y_j - g_{kj}y_h\right) + 2P^i_{kh}y^t\mathcal{B}_t C_{ijl}\ .$$

Using (4.2) in (4.15), we get

(4.16) $$\mathcal{B}_l\left\{\frac{1}{n-1}\left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right)\right\} = \lambda_l\left\{\frac{1}{n-1}\left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right)\right\} + \mu_l\left(g_{kh}y_j - g_{kj}y_h\right) + 2P^i_{kh}y^t\mathcal{B}_t C_{ijl}\ .$$

Transvecting (4.2) by g^{ij}, using (1.9b), we get

(4.17) $$P^i_{kh} = \frac{1}{n-1}\left(P^i h_{kh} + P_k h^i_h + P_h h^i_k\right),$$

Where $P^i = g^{ij}P_j$ and $h^i_h = g^{ij}h_{hj}$

Taking the covariant derivative for (4.17) with respect to x^l in the sense of Berwald and using (2.3) we get

(4.18) $$\mathcal{B}_l\left\{\frac{1}{n-1}\left(P^i h_{kh} + P_k h^i_h + P_h h^i_k\right)\right\} = \lambda_l P^i_{kh} + \mu_l\left(y^i g_{kh} - \delta^i_k y_h\right).$$

Using (4.17) in (4.18), we get

(4.19) $$\mathcal{B}_l\left\{\frac{1}{n-1}\left(P^i h_{kh} + P_k h^i_h + P_h h^i_k\right)\right\} = \lambda_l\left\{\frac{1}{n-1}\left(P^i h_{kh} + P_k h^i_h + P_h h^i_k\right)\right\} + \mu_l\left(y^i g_{kh} - \delta^i_k y_h\right).$$

Show that the tensors $\left\{\frac{1}{n-1}\left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right)\right\}$ and $\left\{\frac{1}{n-1}\left(P^i h_{kh} + P_k h^i_h + P_h h^i_k\right)\right\}$ are given by (4.16) and (4.19), respectively.

Thus, we conclude

Theorem 4.4. *In $P - reducible - G(\mathcal{B}P) - RF_n$, the tensors $\left\{\frac{1}{n-1}\left(P_j h_{kh} + P_k h_{hj} + P_h h_{jk}\right)\right\}$ and $\left\{\frac{1}{n-1}\left(P^i h_{kh} + P_k h^i_h + P_h h^i_k\right)\right\}$ are given by Eqs. (4.16) and (4.19), respectively.*

In $P -\ reducible$ space , we have the following identity [14]

(4.20) $P_{ijkh} + P_{jhki} + P_{hikj} = 0$

Taking the covariant derivative for the left side of (4.20) with respect to x^l in the sense of Berwald and using (2.2), we get

$$\mathcal{B}_l P_{ijkh} + \mathcal{B}_l P_{jhki} + \mathcal{B}_l P_{hikj} = \lambda_l (P_{ijkh} + P_{jhki} + P_{hikj})$$
$$+ \mu_l (g_{ij} g_{kh} - g_{ik} g_{jh} + g_{jh} g_{ki} - g_{jk} g_{hi} + g_{hi} g_{jk} - g_{hk} g_{ij})$$
$$-(P^i_{jkh} + P^i_{jhk} + P^i_{hkj}) \mathcal{B}_l g_{im} .$$

Using (4.20) in above equation and using the symmetric property of metric tensor g_{jh}, we get

(4.21) $\mathcal{B}_l P_{ijkh} + \mathcal{B}_l P_{jhki} + \mathcal{B}_l P_{hikj} = (P^i_{jkh} + P^i_{jhk} + P^i_{hkj}) \mathcal{B}_l g_{im}$

Thus, we conclude

Theorem 4.5. *In $P - reducible - G(\mathcal{B}P) - RF_n$, we have the identity (4.21).*

5. Conclusion

We developed The generalized $\mathcal{B}P -$ recurrent Finsler space by using properties of $P^* -$ space and $P -$ reducible space, we got new spaces which called $P^* -$generalized $\mathcal{B}P -$ recurrent space and $P -$ reducible generalized $\mathcal{B}P -$ recurrent space, respectively. We obtained certain identities in $P^* -$ generalized $\mathcal{B}P -$recurrent space and $P -$ reducible generalized $\mathcal{B}P -$ recurrent space .

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