Some Inverses on Generalized idempotent fuzzy matrices

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***Abstract*—— In this paper, existence and construction of various inverses for the *k* – Idempotent fuzzy matrix were discussed. With this notion, some properties for *k* – idempotent fuzzy matrix also found and we exhibit the relation between Moore – Penrose inverse and group inverse.**

***Keywords*— *k* – idempotent fuzzy matrix, *g* – inverse, Moore – Penrose inverse, Least Square and Minimum norm *g* – inverse, group inverse and Drazin inverse.**

***AMS Subject Classification*—— Primary : 15B15, 17C17 and secondary : 05A05**

I. INTRODUCTION

A fuzzy matrix is a matrix with elements having values in the closed interval [0,1]. The theory of fuzzy matrices has received increasing attention in the past three decade. The concepts of sections of a fuzzy matrix were introduced by Kim and Roush [1] in 1980. Also, Hashimoto [2] studied the canonical form of a transitive fuzzy matrix more specifically max – min transitivity. Even though Kim [5] studies the concepts of idempotent fuzzy matrix in earlier, Hong Youl Lee [3] developed the concepts and gave lot of properties in it which plays a vital role in idempotency of fuzzy matrix. Furthermore, the idempotent fuzzy matrix has generalized by K.Muthugurupackiam et. al. [7] with a special type of matrix namely *k* – idempotent fuzzy matrix. A fuzzy matrix $A = (a_{ij})_{n\times n}$ is said to be *k* – idempotent if, and only if $KA^2K = A$, where K is the permutation matrix, In this paper, various generalized inverses [6] are determined for the *k* – idempotent fuzzy matrix *A*. Kim discusses already about the inverses of fuzzy matrices and the Hashimoto discusses about sub – inverses of fuzzy matrices. A.R.Meenachi [6] introduced the concept of range hermitian (EP matrix) as a generalization of k – hermitian matrices. The g – inverse, Moore – penrose inverse, group inverse and Drazin inverse of the *k* – idempotent fuzzy matrix *A* are denoted as $A_k^-, A_k^+, A_k^\#$ and A_k^d respectively. The set of all g – inverses of *A* is denoted as *A{1}*. Similarly the set of semi inverses, least square *g* – inverses and minimum norm *g* – inverses of A are given by *A{1,2}*, *A{1,3}* and *A{1,4}* respectively. The row space and the column space of A are denoted as $R(A)$ and $\rho(A)$ respectively. Moreover, a fuzzy matrix *A* is regular if and only if there exists a g – inverse of *A* [6].

II. INVERSES ON GENERALIZED IDEMPOTENT FUZZY MATRICES

Definition 2.1

For a *k* – idempotent fuzzy matrix *A*, A_k^- is said to be a *g* – inverse of *A* if $AA_k^-A = A$ for all $A_k^- \in A\{1\}$.

Theorem 2.2

Let *A* be a k – idempotent fuzzy matrix, *X* be a *g* – inverse of *A*, then *AX* and *XA* are *k* – idempotent.

Proof. Consider , $K(AX)^2K = KA^2X^2K$

$$= KA^2K.KX^2K$$
$$= AX$$

Also,

$$K(XA)^2K = KX^2A^2K$$
$$= KX^2K.KA^2K$$
$$= XA$$

Hence, the fuzzy matrices AX, XA are k – idempotent.

Example 2.3

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, then there exists a g – inverse of A, $X = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ such that $AXA = A$

Here, $AX = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $XA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

and $K(AX)^2K = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $K(XA)^2K = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

Therefore, AX and XA are k – idempotent.

Lemma 2.4

For $A \in f_{mn}$, A is a regular fuzzy matrix $\Leftrightarrow$ there exists an k – idempotent fuzzy matrix H such that $R(H) = R(A)$.

Lemma 2.5

For $A \in f_{mn}$, A is a regular fuzzy matrix $\Leftrightarrow$ there exists an k – idempotent fuzzy matrix H such that $\rho(H) = \rho(A)$.

Theorem 2.6

If A_k^- is a g – inverse of a k – idempotent fuzzy matrix A, then g – inverse A^2 is KA_k^-K.

Proof.

$$A^2(KA_k^-K)A^2 = A^2KA_k^-KA^2$$
$$= KAA_k^-AK$$
$$= KAK$$
$$= A^2$$

Definition 2.7

For a k – idempotent fuzzy matrix A, there exists a semi – inverse X for A such that $AXA = A$ and $XAX = X$ for all $X \in A\{1,2\}$.

Theorem 2.8

If A is a k – idempotent fuzzy matrix, then A^2 is both g – inverse and semi – inverse of A.

Proof.

$$AA^2A = A^4$$
$$= A \quad \{ \text{ since A is quadrapotent}$$
$$\therefore \quad A^2 \in A\{1\}$$
And $$A^2AA^2 = A^5$$
$$= A$$

$$\therefore \quad A^2 \in A\{1,2\}$$

III. MOORE – PENROSE INVERSE OF A K – IDEMPOTENT FUZZY MATRICES

Definition 3.1

For a k – idempotent fuzzy matrix A, a matrix G is said to be least square g – inverse of A if $AGA = A$ and $(AG)^T = AG$ for all $X \in A\{1,3\}$.

Definition 3.2

For a k – idempotent fuzzy matrix A, a matrix G is said to be minimum norm g – inverse of A if $AGA = A$ and $(GA)^T = GA$ for all $X \in A\{1,3\}$.

By the theorem 2.6, we will get the following results obviously.

Theorem 3.3

If $A_k{}^{(1,3)}$ is a $\{1,3\}$ – inverse of a k – idempotent fuzzy matrix A, then

$$[A^2]_k{}^{(1,3)} = KA_k{}^{(1,3)}K.$$

Theorem 3.4

If $A^{(1,4)}$ is a $\{1,4\}$ – inverse of a k – idempotent fuzzy matrix A, then

$$[A^2]_k{}^{(1,4)} = KA_k{}^{(1,4)}K.$$

Definition 3.5

For a k – idempotent fuzzy matrix A, if the fuzzy matrix G is both $\{1,3\}$ – inverse and $\{1,4\}$ – inverse, then G is said to be the Moore – Penrose inverse of A, and is denoted as $A_k{}^{+}$.

By combining the theorems 2.6, 3.3 and 3.4, we easily prove the following theorem.

Theorem 3.6

If $A_k{}^{+}$ is a Moore Penrose inverse of a k – idempotent fuzzy matrix A, then the Moore Penrose inverse of A^2 is $K A_k{}^{+} K$.

Theorem 3.7

Let A be a k – idempotent fuzzy matrix, then $(A^3)_k{}^{+}$ commutes with the associated permutation matrix K.

Proof.

If A is a k – idempotent fuzzy matrix, then by $KA = AK$ [2]

$$KA^3 = A^3K$$
$$(KA^3)_k{}^{+} = (A^3K)_k{}^{+}$$
$$K_k{}^{+}(A^3)_k{}^{+} = (A^3)_k{}^{+}K_k{}^{+}$$
$$K(A^3)_k{}^{+} = (A^3)_k{}^{+}K$$

Theorem 3.8

Let A be a k – idempotent fuzzy matrix then $A_k{}^{+}$ is also k – idempotent fuzzy matrix.

Proof.

$$A^2(A_k{}^+)^2A^2 = (AA)(A_k{}^+A_k{}^+)(AA)$$
$$= (AAA_k{}^+)(A_k{}^+AA)$$
$$= (AA_k{}^+A)(AA_k{}^+A)$$
$$= AA$$
$$= A \quad (3.1)$$

and $(A_k{}^+)^2A^2(A_k{}^+)^2 = (A_k{}^+A_k{}^+)(AA)(A_k{}^+A_k{}^+)$

$$= (A_k{}^+A_k{}^+A)(AA_k{}^+A_k{}^+)$$
$$= (A_k{}^+AA_k{}^+)(A_k{}^+AA_k{}^+)$$
$$= A_k{}^+A_k{}^+$$
$$= (A_k{}^+)^2 \quad (3.2)$$

From (3.1) and (3.2), we have

$$(A^2)_k{}^+ = (A_k{}^+)^2.$$

Now
$$A_k{}^+ = (KA^2K)_k{}^+$$
$$= K(A^2)_k{}^+K$$
$$= K(A_k{}^+)^2K$$

Hence $A_k{}^+$ is k – idempotent fuzzy matrix.

IV. SPECTRAL INVERSES

Definition 4.1

For a k – idempotent fuzzy matrix A, the group inverse of A, denoted as $A_k{}^\#$ is a commuting semi – inverse of A, that is, $AA_k{}^\#A = A$, $A_k{}^\#AA_k{}^\# = A_k{}^\#$ and $AA_k{}^\# = A_k{}^\#A$.

Theorem 4.2

If $A_k{}^\#$ is a group inverse of a k – idempotent fuzzy matrix A, then the group inverse of A^2 is $KA_k{}^\#K$.

Proof.

$$A^2(KA_k{}^\#K)A^2 = A^2KA_k{}^\#KA^2$$
$$= KAA_k{}^\#AK$$
$$= KAK$$
$$= A^2$$

In the similar manner, we will prove

$$(KA_k{}^\#K)A^2(KA_k{}^\#K) = KA_k{}^\#K$$

Next, by the definition, $A = A^2A_k{}^\# = A_k{}^\#A^2$

Multiplying K^2 on both sides, $K^2(A_k{}^{\#}A^2) = (A_k{}^{\#}A^2)K^2$

That is, $(KA_k{}^{\#}K)A^2 = A^2(KA_k{}^{\#}K)$

Hence $KA_k{}^{\#}K$ is the group inverse *of* A^2

Lemma 4.3

If $A_k{}^{\#}$ is a group inverse of a k – idempotent fuzzy matrix A, then $A^2 = A_k{}^{\#}$.

Proof. Since by theorem 2.10, $A^2 \in A\{1,2\}$

for the k – idempotent fuzzy matrix $A = A^3 = AA^2 = A^2A$.

Then we have $A_k{}^{\#} = A^2$.

Example 4.4

Let $A = \begin{pmatrix} 0.1 & 0.1 & 0.1 \\ 0.2 & 0.4 & 0.2 \\ 0.7 & 0.6 & 0.8 \end{pmatrix}$. Since $KA^2K = A$, A is idempotent.

Hence $A_k{}^{\#}$ exists and $A_k{}^{\#} = A$.

Here $AA^TA \neq A \Rightarrow A^T$ is not a g – inverse of A.

Thus $A_k{}^{+}$ does not exist.

Hence for a k – idempotent fuzzy matrix A, if $A_k{}^{\#}$ exists then $A_k{}^{+}$ need not exist.

Remark 4.5

A k – idempotent fuzzy matrix A is said to be EP if and only if $A_k{}^{+} = A_k{}^{\#}$.

Theorem 4.6

Let a k – idempotent matrix A is EP then KA and AK are also EP.

Proof.

If A is EP then $A_k{}^{+} = A_k{}^{\#}$

i.e., $A_k{}^{+} = A^2$

i.e., $A_k{}^{+} = KAK$

i.e., $A_k{}^{+}K = KA$

i.e., $(KA)_k{}^{+} = (KA)_k{}^{\#}$

in the similar way, we prove $(AK)_k{}^{+} = (AK)_k{}^{\#}$.

Hence AK and KA are EP.

Definition 4.7

For a k – idempotent fuzzy matrix A, the Drazin inverse of A denoted as $A_k{}^{d}$ is solution of the following equations:

For some positive integer m, $A^m = A^{m+1}$

$$X = X^2A$$
$$AX = XA.$$

Theorem 4.8 If ${A_k}^d$ is the Drazin inverse of k – idempotent fuzzy matrix A then ${A_k}^d = A^2$

Proof. $A = A^4 = A^2 A^2$. Hence $m = 1$

$A^2 = A^5 = (A^2)^2 A$

and $AA^2 = A^3 = A^2 A$

Hence ${A_k}^d = A^2$.

V. CONCLUSION

With the existence of the inverse of k – idempotent fuzzy matrix, the row space and column space for them and the retract were discussed. It has found that the square of a k – idempotent fuzzy matrix behaves as its inverse. Also, it has discussed that some of the inverse of a k – idempotent fuzzy matrix is also a k – idempotent fuzzy matrix. The Moore Penrose inverse has constructed by combining minimum norm and least square g – inverse. Since the application of g – inverses are rapidly increasing now a days, we expect that these fundamental properties will be gain much importance in fuzzy set theory.

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