A Note on Exact Solution of certain linear and non linear Partial Differential Equations By Homotopy Perturbation Method

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Abstract: A new homotopy technique is introduced to find the solution of first order in homogeneous Partial Differential Equation $v_x(x,y) + \alpha(x,y)v_y(x,y) + \beta(x,y)h(v) = f(x,y)$. The Homotopy Perturbation method is combined with the decomposition of a source function to develop this technique. In this technique, the homotopy constructed is based on the decomposition of a source function. Decomposition of source function in various ways leads to different homotopies. Also by using the fact that the convergence of a solution is being affected by the decomposition of source function, which leads to develop a new method to decompose a source function that accelerates the convergence of a solution. The purpose of this study is to establish the fact that by constructing the proper homotopy through decomposing $f(x,y)$ in a proper way which determines the solution with less computational work and reliable results. This method can be generalized to all inhomogeneous partial differential equation problems.

Key-Words: Mitting-Leffler functions, Fokker-Plank equation, Exact solutions, biological popuations, Caputo derivative

1. INTRODUCTION

In recent years may problems in [1,2] the fields of Physics, Chemistry and Engineering are modeld by linear and non-linear PDEs, in which Homotopy Perturbation method is employed extensively to find the solution especially boundary and initial value problems. This method was first introduced by Ji Huan He, [3,4,5] is a powerful mathematical tool which helps to investigate a variety of problems arising in many fields. This method was obtained by combining homotopy theory in topology with perturbation theory. In this method, the different problems under study is continuously decomposed into a simple problem, so as to find an analytic or approximate solution .In this paper, a new technique is proposed to find the solution $v(x,y)$ based on the decomposition of a right hand side function $f(x,y)$, through which new homotopies are constructed. It is very clearly shown that by decomposing $f\ (x,y)$ in a proper way, computational work will be less than the existing approach to determining the solution. Thos method is very effective and simple.

The paper is organized as below: In sec 2, new homotopy perturbation method is analyzed. Sec 3 is devoted for constructing a new homotopy based on the decomposition of the source function for a first order inhomogeneous PDE. In sec 4, the method is analyzed for linear problems. Some illustrations are given in sec 5. Finally sec 6 is given for concluding remarks.

2. HOMOTOPY PERTURBATION METHOD

To illustrate basic ideas of the homotopy analysis method, let us consider the following non-linear differential equations:

$$A(v) - f(r) = 0\ ,\ r \in \Omega \tag{1}$$

with the boundary conditions

$$B\left(v, \frac{\partial v}{\partial r}\right) = 0\ ,\ r \in \Gamma \tag{2}$$

Where A is a general differential operator, B is a boundary operators $f(r)$ *is* an analytic, function, and Γ is the boundary of domain Ω .

The operator A in (1) can be rewritten as a sum of L and N, L and N are linear and nonlinear parts of A, respectively, as follows:

$$L(v) + N(v) - f(r) = o$$

By the homotopy technique, we construct the following homotopy

$$H(u,p) = (1-p)\big(L(u) - L(v_0)\big) + p(A(u) - f(r) = 0 \quad (3)$$

Which is equivalent to

$$H(u,p) = L(u) - L(v_0) + pL(v_0) + p(N(u) - f(r)) = 0 \quad (4)$$

Where

$u(r,p): \Omega\ x\ [0,1] \rightarrow \Re$,

$p \in [0,1], r \in \Omega$,

p is an embedding parameter, v_0 is an initial approximation of (1) which satisfies the boundary conditions. As p changes fron zero to unity $u(r,p)$ changes from v_0 to $v(r)$. Here the convergence of a solution depends on the choice of v_0 , that is , we can have different approximation solutions for different v_0 .

Let us decompose the source function as $f(r) = f_1(r) + f_2(r)$.

$$L(v_0) = f_1(r)$$

In (3) we get the following homotopy:

$$H(u,p) = (1-p)\big(L(u) - f_1(r)\big) + p\big(A(u) - f(r)\big) = 0 \quad (5)$$

Which is equivalent to

$$H(u,p) = L(u) - f_1(r) + p(N(u) - f_2(r)) = 0 \quad (6)$$

Obviously, from (6) we get

$$H(u,0) = L(u) - f_1(r) = 0$$

$$H(u,1) = A(u) - f(r) = 0$$

As the embedding parameter p changes from zero to unity, $u(r,p)$ changes from $L^{-1}\big(f_1(r)\big)$ to $v(r)$. According to He's homotopy perturbation method, we can first use the embedding parameter p As a small parameter and assume that the solution of (6) can be written as a power series in p as follows:

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots \quad (7)$$

Setting $p = 1$, we get the approximate solution of (1)

$$v = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \cdots$$

In the next section, let us go for the decomposition of the right hand side function which has a great effect on the amount of calculation and the speed convergence of the solution.

3. CONSTRUCTION OF A NEW HOMOTOPY BASED ON THE DECOMPOSITION OF A SOURCE FUNCTION

Let us consider the following boundary value problem with the following inhomogeneous PDE.

$$v_x(x,y) + a(x,y)v_y(x,y) + b(x,y)h(v) = f(x,y), \quad (8)$$

$$v(0,y) = g(y). \quad (9)$$

Where a,b,h and f are continuous functions in some of the plane and $g(0) = 0$.

By solving this boundary value problem by the homotopy perturbation method, we get an approximate or exact solution *u(x,y)*. Before proceeding further, let us introduce the integral operator I defined in the following form:

$$I\,(.) = \int_0^x (.)dx \tag{10}$$

Then the derivative of operator I with respect to y is defined as

$$I_y(.) = \frac{\partial}{\partial y}\int_0^x (.)dx$$

Rewriting *f(x,y)* as a sum of two functions $f(x,y) = f_1(x,y) + f_2(x,y)$ and constructing a homotopy in equation (6) , we have

$$H(u,p) = (1-p)(u_x + f_1) + p(u_x + a(x,y)u_y + b(x,y)h(u) - f) \tag{11}$$

Which is equivalent to

$$H(u,p) = u_x(x,y) - f_1(x,y) + p(a(x,y)u_y(x,y) + b(x,y)h\big(u(x,y)\big) - f_2(x,y)) = 0 \tag{12}$$

Substituting (7) into (12) & equating the coefficient of the terms by the same power in p , we have

$$p^0: \big(u_0(x,y)\big)_x - f_1(x,y) = 0,\;\; u_0(o,y) = h(y) \Rightarrow u_0(x,y) = I(f_1(x,y) + g(y),$$

$$p^1\big(u_1(x,y)\big)_x + a(x,y)\big(u_0(x,y)\big)_y + b(x,y)h\big(u_0(x,y)\big) - f_2(x,y) = 0 \;\; u_1(0,y) = 0 \Rightarrow$$

$$(u_1)_x + a(x,y)\big[I_y(f_1) + g_y(y)\big] + b(x,y)h\big(I(f_1) + g(y)\big) - f_2(x,y) = 0 \tag{13}$$

If there exists the relationship between $f_1(x,y)$ & $f_2(x,y)$,

$$a(x,y)\big[I_y\big(f_1(x,y)\big) + g_y(y)\big] + b(x,y)h\Big(I\big(f_1(x,y)\big) + g(y)\Big) = f_2(x,y) \tag{14}$$

then we have from (13)

$$(u_1)_x = 0 \qquad u_1(o,y) = 0 \qquad \Rightarrow \quad u_1 = 0$$

and

$$p^2: (u_2)_x + a(u_1)_y + bh(u_1) = 0 \qquad u_2(0,y) = 0 \qquad \Rightarrow \quad u_2 = 0,$$

$$p^3: (u_3)_x + a(u_2)_y + bh(u_2) = 0 \qquad u_3(0,y) = 0 \qquad \Rightarrow \quad u_3 = 0,$$

Hence the approximate or exact solution of problem (8)-(9) is obtained as follows:

$$v(x,y) = u_0(x,y),$$

And the source function is obtained from (14) which of the form :

$$f(x,y) = f_1 + a(x,y)\big[I_y(f_1) + g_y(y)\big] + b(x,y)h\big(I(f_1) + h(y)\big). \tag{15}$$

Equation (15) will be the required condition to accelerate convergence. If that equation has a solution, then we may get the approximate or exact solution of problem (8)-(9) in two steps. But it is not always possible to decompose the source function $f(x,y)$ in such a way that the functions $f_1(x,y)$ and $f_2(x,y)$ have the relationship in (14) . In that case then we are looking for an arbitrary $s\,(x,y)$ such that the functions $f_1(x,y)$ and $f_2(x,y)$ have the following relationship.

$$a(x,y)\big[I_y(f_1) + g_y(y)\big] + b(x,y)h(I(f_1) + g(y)) = f_2 + S(x,y) \tag{16}$$

Then we get the approximate or exact solution of the problem in more than two steps. And also we can get the solution in the form of series.

4.NEW HOMOTOPY PERTURBATION ANALYSIS FOR LINEAR PROBLEMS

In this section we demonstrate the theory of the new homotopy perturbation method given in section 2 and 3 for some special $f(x,y)$ in the following linear problems:

$$v_x(x,y) + av_y(x,y) + bv(x,y) = f(x,y), \tag{17}$$
$$v(0,y) = h(y), \tag{18}$$

Where *a* and *b* are constants

4.1 Case 1: Source function *f(x,y)* is a polynomial

We consider that $g(y) = 0$ and $f(x,y)$ is an *n*th- order polynomial in problem (17) –(18). Hence, $f(x,y)$ can be written in the following form.

$$f(x,y) = \sum_{i=0}^{n} \sum_{\alpha+\beta=i} K_{\alpha\beta}\, x^{\alpha}\, y^{\beta} \tag{19}$$

Where $k_{\alpha\beta}$ are constant and α, β are natural numbers.

The polynomial $f(x,y)$ may be decomposed as

$$f(x,y) = f_1(x,y) + a\, I_y\big(f_1(x,y)\big) + bI)f_1(x,y)) \tag{20}$$

From (15), we get the solution of the problem in the following two steps. We take $f_1(x,y)$ as an *n*th- order Polynomial which may be given as

$$f_1(x,y) = \sum_{i=0}^{n} \sum_{\alpha+\beta=i} A_{\alpha\beta}\, x^{\alpha}\, y^{\beta} \tag{21}$$

Where $A_{\alpha\beta}$ are constants then $I(f_1(x,y))$ becomes

$$I(f_1(x,y) = \sum_{i=0}^{n} \sum_{\alpha+\beta=i} \frac{A_{\alpha\beta}}{\alpha+1}\, x^{\alpha+1}\, y^{\beta} \tag{22}$$

And $I_y(f_1(x,y))$ becomes

$$I_y f_1(x,y)) = \sum_{i=0}^{n} \sum_{\substack{\alpha+\beta=i \\ \beta>1}} \frac{\beta A_{\alpha\beta}}{\alpha+1} x^{\alpha+1}\, y^{\beta-1} \tag{23}$$

In order to determine the unknown coefficients $A_{\alpha\beta}$ in terms of $K_{\alpha\beta}$, we substitute (21) ,(22), and (23) into (20).

$$f(x,y) \sum_{i=0}^{n} \sum_{\alpha+\beta=i} A_{\alpha\beta}\, x^{\alpha}\, y^{\beta} + \sum_{i=0}^{n} \sum_{\substack{\alpha+\beta=i \\ \beta>1}} \frac{\beta a A_{\alpha\beta}}{\alpha+1} x^{\alpha+1}\, y^{\beta-1}$$
$$+ \sum_{i=0}^{n} \sum_{\substack{\alpha+\beta=i \\ \beta>1}} \frac{b A_{\alpha\beta}}{\alpha+1} x^{\alpha+1}\, y^{\beta} \tag{24}$$

and then using (19), we get the following

$k_{00} = A_{oo}$

$K_{10} = A_{10} + bA_{00}$

$K_{01} = A_{01}$

$K_{20} = A_{20} + \frac{b}{2} A_{10}$

$K_{02} = A_{02},$

$K_{11} = A_{11} + 2aA_{02} + bA_{01}$

$\vdots$

Then we obtain

$A_{00} = K_{00}$

$A_{10} = K_{10} - bK_{00}$

$A_{01} = K_{01}$

$A_{20} = K_{20} - \frac{b}{2} (K_{10} - bK_{00})$

$A_{02} = K_{02}$

$A_{11} = K_{11} - 2aK_{02} - bK_{01}$

$\vdots$

As a result, we can construct the following homotopy :

$H(u,p) = (1-p)\big(u_x - f_1(x,y)\big) + p\big(u_x + au_y + bu - f(x,y)\big) = 0$

Which lead us to arrive the solution in two steps

4.2 Case 2: The source function *f(x,y)* is a sum of two functions *m(x)* and *n(y):*

Let us take $f(x,y) = m(x) + n(y)$ in problem (17) – (18). Since *f(x,y)* is a continuous function, *m(x)* and *n(y)* are also continuous functions. Based on (15), *f(x,y) = m(x) + n(y)* is decomposed as follows:

$$f(x,y) = m_1(x) + n_1(y) + a\,I_y\big(m_1(x) + n_1(y)\big) + ag_y + bI(m_1(x) + n_1(y)) + bg(y) \qquad (25)$$

Then we can get the solution of the problem in the following two steps. Hence the function *f (x,y)* can be rewritten in the following form.

$$f(x,y) = m_1(x) + n_1(y) + an_{1y}(y)x + ag_y(y) + b\int_0^x m_1(x)dx + bn_1(y)x + bg(y) \qquad (26)$$

Since $f(x,y) = m(x) + n(y)$, we have

$$an_{1y}(y)x + bn_1(y)x = 0 \quad \text{for } x \neq 0 \qquad (27)$$

The solution of the above ordinary differential equations becomes

$$n_1(y) = 0 \qquad (28)$$

and

$$n_1(y) = e^{-\frac{b}{a}y} \qquad (29)$$

On the other side, we have

$$n(y) = ag_y(y) + bg(y) + n_1(y) \qquad (30)$$

and

$$m(x) = b\int_0^x m_1(x)dx + m_1(x) \qquad (31)$$

If either (28) or (29) satisfy the equation (30) & equation (31) has a solution, then the decomposition of the function *f (x,y)* is possible to accelerate the convergence of the solution.

As a result by constructing the following homotopy:

$$H(u,p) = (1-p)(u_x - m_1(x) - n_1(y)) + p(u_x + au_y + bu - m(x) - n(y)) = 0$$

We achieve the solution in two steps.

5. NUMERICAL ILLUSTRATION

(a) Consider the inhomogeneous linear boundary value problems with constant coefficients.

$$v_x - v_y + v = e^y + e^x, \qquad v(0,y) = e^y \tag{32}$$

By constructing the following homotopy

$$H(u,p) = (1-p)(u_x - m_1(x) - n_1(y)) + p\left(u_x - u_y + u - e^x - e^y\right) = 0 \tag{33}$$

Which is equivalent to

$$H(u,p) = u_x - m_1(x) - n_1(y) + p\left(-u_y + u + m_1(x) + n_1(y) - e^x - e^y\right) = 0 \tag{34}$$

We get the function $m_1(x), n_1(y)$ as it is explained in section 4.2. It is obvious that $n_1(y) = e^y$ satisfies equation (30). Moreover, from equations (31) $m_1(x)$ can be found as follows:

$$m_1(x) = \frac{e^x + e^{-x}}{2} \tag{35}$$

If we put the function $m_1(x)$ and $n_1(y)$ in the homotopy (33) or (34), then we get

$$H(u,p) = u_x - \frac{e^x + e^{-x}}{2} - e^y + p\left(-u_y + u + \frac{e^{-x} - e^x}{2}\right) = 0 \tag{36}$$

And we get the following

$$p^0: (u_0)_x - \frac{e^x + e^{-x}}{2} - e^y = 0 \implies u_0 = \frac{e^x - e^{-x}}{2} + e^y x + e^y,$$

$$p^1: (u_1)_x - (u_0)_y + u_0 + \frac{e^{-x} - e^x}{2} = 0 \implies u_1 = 0$$

$$p^2: (u_2)_x - (u_1)_y + u_1 = 0 \implies u_1 = 0$$

$\vdots$

Hence the solution *u(x,y)* becomes,

$$v(x,y) = u_0 = \frac{e^x - e^x}{2} + e^y x + e^y \tag{37}$$

Which is the exact solution of the problem .

(b) Consider the following inhomogeneous linear boundary value problem with variable coefficients:

$$yv_y - v + v_x = 2xy^2 + 2y^2, \ v(0,y) = 0 \tag{38}$$

In this problem the right hand side function *f (x,y)* is a third order polynomial. Then from (15) we have

$$f(x,y) = f_1(x,y) + yI_y\left(f_1(x,y)\right) - I(f_1(x,y)) \tag{39}$$

Based on the source function $f(x,y) = 2xy^2 + 2y^2$ if we take $f_1(x,y)$ as

$$f_1(x,y) = c_1x^3 + c_2x^2y + c_3xy^2 + c_4y^3 + c_5x^2 + c_6xy + c_7y^2 + c_8x + c_9y + c_0 \tag{40}$$

And substitute (40) into (39), we get

$$c_7 = 2$$

$$c_1 = c_2 = c_3 = c_4 = c_{5+}c_6 = c_8 = c_9 = c_0 = 0$$

Based on the result, we get $f_1(x,y) = 2y^2, f_2(x,y) = 2y^2$ and the homtopy will be Constructed in the following form:

$$H(u,p) = u_x - 2y^2 + p\left(yu_y - u - 2xy^2\right) = 0 \qquad (41)$$

and we get the following

$p^0: (u_0)_x - 2y^2 = 0 \implies u_0 = 2xy^2,$

$p^1: (u_1)_x + y(u_0)_y - u_0 - 2xy^2 = 0 \implies u_1 = 0$

Then, we obtain the exact solution in the following form

$$v(x,y) = u_0 = 2xy^2 \qquad (42)\ .$$

(c) Let us consider the following B V P

$$v_x + v_y - xv = ye^{xy} + x \qquad (43)$$

$v(0,y) = 0$

By using (15) we obtain

$$ye^{xy} + x = f_1 + I_y(f_1) - xI(f_1) \qquad (44)$$

Based on the source function $f(x,y) = ye^{xy} + x,$ we come to a conclusion that $f_1(x,y)$ is in the following form :

$$f_1(x,y) = c_1e^{xy} + c_2xe^{xy} + c_3x \qquad (45)$$

Substituting (45) into (44) we get

$c_1 = c_2 = 0$

$c_3 = 1$

Now, let us take $f_1(x,y) = ye^{xy}, f_2(x,y) = x$ and construct the homotopy I the following form:

$$H(u,p) = u_x - ye^{xy} + p\left(u_y - xu - x\right) = 0 \qquad (46)$$

Which gives the following equations:

$p^0: (u_0)_x - ye^{xy} = 0 \implies u_0 = e^{xy} - 1,$

$p^1: (u_1)_x + (u_0)_y - xu_0 - x = 0 \implies u_1 = 0$

Hence, the exact solution becomes

$$v(x,y) = u_0 = e^{xy} - 1 \qquad (47)$$

Which dealt with minimum of calculations.

6. CONCLUSION

In this paper we applied a new homotopy perturbation method to find the solution of first order inhomogeneous partial differential equations. Here, every decomposition of source function *f(x,y)* will help to arrive a new homotopy. We developed a method to obtain the proper decomposition of *f(x,y)* though which, we can obtain the solution with minimum computation and accelerate the convergence of the solution. This method proves that the decomposition of the source function has a great effect towards the amount of computations and the acceleration of the convergence of the solutions. When compared to the standard one, the decomposition of the source function *f(x,y)* in a proper way, will lead to a simple and very effective tool for calculating the exact or approximate solution with less computational work.

REFERENCES

(1) *Junfeng. L An analytical approach to the sine- Gordon equation using the modified homotopy Perturbation method, Comput Math Appl. 58 2313-2319 (2009)*

(2) *Neamaty, A. Darzi, T. Comparison between the variational iteration method and the homotopy perturbation method for the Sturm-Liouville differential equation, Bound Value Propl 2010 317369 (2010)*

(3) *He, JH. Homotopy perturbation technique, Comput Methods Appl. Mech Eng. 178,257-262 (1999)*

(4) *He ,JH. A coupling method of homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-Linear Mech35,37-43 (2000).*

(5) *He, JH. Homotopy perturbation method a new nonlinear analytical technique, Appl . Mech.Comput 135,73-79 (2003)*

(6) *He,JH. Comparison of homotopy method and homotopy analysis method, Appl. Mech. Comput 156,527-539 (2004)*