Regular f-Derivations on BH algebras.

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ABSTRACT

Motivated by some results on derivations on rings and the generalizations of BCK and BCI algebras, in this paper, we define f-derivations on BH-algebras and investigate some important results.

Key Words: BH-algebras, derivations on BH-algebras, f -derivations and regular f –derivations on BH-algebras.

1.Introduction:

BCK and BCI algebras are two new classes of algebras based on propositional calcului or logic introduced by Imai and Isaki[1].In[3] K.Isaki and K.Tanaka introducd the theory of BCK-algebras. In[2] Q.P.Hu and X.Li and introduced a wider class of abstract algebras BCH-algebras. The class of BCI-algebras is a proper subclass of the class BCH algebras. J.Neggers and H.S.Kim [7] introduced the notion of d-algebras which is another generalization of BCK-algebras

In 2004 Y.B.Jun and X.L..Xin [5] inroduced the notion of derivations of BCI-algebras, which was motivated from a lot of workdone on derivations of rings. Since then many authors worked on the notion of derivations on several algebras such as d-algebras[8] and TM-algebras motivated by this paper introduce the notion of f-derivations on BH-algebras.

2. Preliminaries

Definition 2.1: [2]Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is called a **BCK-algebras** if it satisfies the following axioms:

1. $x * x = 0$
2. $0 * x = 0$
3. $\left((x * y) * (x * z)\right) * (z * y) = 0$
4. $\left(x * (x * y)\right) * y) = 0$
5. $x * y = 0 \text{ and } y * x = 0 \text{ imply } x = y \; \forall x, y, z \in X.$

Definition 2.2: [3]Let X be a set with a binary operation $*$ and a constant 0. Then $(X, *, 0)$ is a called a ***BCI* -algebras** if it satisfies the following axioms:

1. $((x * y) * (x * z)) * (z * y) = 0$
2. $(x * (x * y)) * y = 0$
3. $x * x = 0$
4. $x * y = 0$ and $y * x = 0$ imply $x = y$ $\forall x, y, z \in X$.

Definition 2.3: [6] A ***d* -algebra** is a non-empty set X with a constant 0 and a binary operation $*$ satisfying the following axioms:

1. $x * x = 0$

2. $0 * x = 0$

3. $x * y = 0$ and $y * x = 0 \Longrightarrow x = y$

Definition 2.4: Let X be a BCI -algebra. Two elements x and y in X are said to be **comparable** if $x \leq y$ or $y \leq x$. Here $x \leq y$ if and only if $x * y = 0$. Also we denote $y * (y * x)$ by $x \wedge y$.

Definition 2.5: Let X be a BCI -algebra. By a **(*l*, *r*) − derivation** of X, we define a self map θ of X satisfying the identity

$$\theta(x * y) = (\theta(x) * y) \wedge (x * \theta(y)) \; \forall x, y \in X .$$

If X satisfies the identity

$$\theta(x * y) = (x * \theta(y)) \wedge (\theta(x) * y), \forall x, y \in X ,$$

then we say that θ is a **(*r*, *l*)− derivation** of the BCI-algebra X .

Definition 2.6: [5]Let X be a set with a binary operation $*$ and a constant 0.Then $(X, *, 0)$ is called a **BH-algebra** if it satisfies the following axioms:

1. $x * x = 0$
2. $x * 0 = x$
3. $x * y = 0$ *and* $y * x = 0$ which implies $x = y$ $\forall$ $x, y \in X$.

Example 2.7: Let $X = \{0,1,2\}$ be a set with the following cayley table

$*$	0	1	2
0	0	1	2
1	1	0	2
2	2	2	0

Definition 2.8: Let S be a non-empty subset of a BH-algebra X, then S is called **BH- Subalgebra** of X if $x * y \in S$ $\forall$ $x, y \in S$.

Definition 2.9: Let X be a BH-algebra and I ($\neq 0$) $\subseteq X$. I is called an **BH-ideal** of X if it satisfies: $\forall$ $x, y \in X$

1. $0 \in I$
2. $x * y \in I$ and $y \in I$ imply $x \in I$.

Obviously, $\{0\}$ and X are ideals of X. We will call $\{0\}$ and X a **zero ideal** and a **trivial ideal**, respectively. An ideal I said to be **proper** if $I \neq X$.

Definition 2.10: Let X be a BH -algebra. Define a binary relation $\leq$ on X by taking $x \leq y$ if and only if $x * y = 0$. In this case $(X,\leq)$ is a **partially ordered set.**

Definition 2.11: Let $(X,*,0)$ be a BH-algebra. A mapping f of a BH-algebra X into itself is called an **endomorphism** if $f(x * y) = f(x) * f(y)$.

Definition 2.12: Let X be a BH-algebra. A map θ : X→X is a **left-right derivation of X** (briefly (l, r) derivation), if it satisfies the identity

$$\theta(x*y)=(\theta(x) * y) \wedge (x * \theta(y)), \forall\, x, y \in X.$$

If θ satisfies the identity

$$\theta(x*y)=(x * \theta(y)) \wedge (\theta(x) * y), \forall\, x, y \in X,$$

then θ is a **right-left derivation of X** (briefly (r, l) derivation)**.**

If θ is both a (l, r) and (r, l)- derivation, then θ is a derivation of X.

3. f - DERIVATIONS ON BH-ALGEBRAS

In this section, we define the notion of f-derivations and regular of f-derivations on BH-algebras and prove some results. Throughout this section we assume that f is an endomorphism of the BH-algebra $(X, *, 0)$

Definition3.1: Let X be a BH - algebra. By a left - right f - derivation (briefly, $(l, r) - f$ derivation) of X, we mean a self map θ_f of X satisfying the identity

$$\theta_f(x * y) = \left(\theta_f(x) * f(y)\right) \wedge \left(f(x) * \theta_f(y)\right), \text{ for all } x, y \in X .$$

If θ_f satisfies the identity

$$\theta_f(x * y) = \left(f(x) * \theta_f(y)\right) \wedge (\theta_f(x) * f(y)), \text{ for all } x, y \in X .$$

then it is said that θ_f is a right - left f - derivation (briefly, $(r,l)-f$- derivation) of X.

If θ_f is both an (r,l) and an $(l,r)-f$ derivation, then θ_f is said to be a f - derivation.

Example 3.2: Let X = {0, 1, 2, 3, 4, 5} be a BH - algebra with the following cayley table:

*	0	1	2	3	4	5
0	0	0	0	0	0	0
1	1	0	0	0	0	0
2	2	2	0	0	0	0
3	3	3	1	0	0	0
4	4	2	1	1	0	0
5	5	5	1	1	1	0

Define a map $\theta_f : X \to X$ by $\theta_f(x) = \begin{cases} 0 \ \ if\ x = 0,1,3,5 \\ \ 2 \ \ if\ x = 2,4 \end{cases}$ and define an endomorphism f of X by $f(x) = \begin{cases} 0\ if\ x = 0,1 \\ 2 \ \ otherwise \end{cases}$.

Then it is easily checked that θ_f is both derivation and f- derivation of X.

Example3.3: Let X = {0, 1, 2, 3} be a BH - algebra with the following Cayley table:

*	0	1	2	3
0	0	0	0	0
1	1	0	1	0
2	2	2	0	0
3	3	2	1	0

Define a map $\theta_f : X \to X$ by $\theta_f(x) = \begin{cases} 0 \ \ if\ x = 0,2 \\ 1\ if\ x = 1,3 \end{cases}$ and define an endomorphism f of X by $f(x) = \begin{cases} 0\ if\ x = 0,2 \\ 3 \ \ if\ x = 1,3 \end{cases}$.

Then it is easily checked that θ_f is f- derivation of X.

4. REGULAR f- DERIVATIONS ON BH-ALGEBRAS

Definition 4.1: An f- derivation θ_f of a BH-algebra X is said to be regular if $\theta_f(0) = 0$.

Definition 4.2:Let be a BH-algebra and θ_f be a f-derivation of X. Define $Ker\ \theta_f = \{x \in X / \theta_f(x) = 0\}$ for all $x \in X$.

Proposition 4.3: Every (r, l)-f-derivation of a BH- algebra is regular.

Proof:

Let θ_f be a (r, l)-f-derivation of X. Then for all $x \in X$, we have

$$\begin{aligned}\theta_f(0) = \theta_f(0 * x) &= \left(f(0) * \theta_f(x)\right) \wedge \left(\theta_f(0) * f(x)\right)\\ &= \left(0 * \theta_f(x)\right) \wedge \left(\theta_f(0) * f(x)\right)\\ &= 0 \wedge \left(\theta_f(0) * f(x)\right)\\ &= \left(\theta_f(0) * f(x)\right) * \left(\theta_f(0) * f(x)\right) * 0)\\ &= \left(\theta_f(0) * f(x)\right) * \left(\theta_f(0) * f(x)\right)\\ &= 0\end{aligned}$$

Proposition 4.4: Every (l, r)-f-derivation of a BH- algebra is regular.

Proof:

Let θ_f be a (l, r)-f-derivation of X. Then for all $x \in X$, we have

$$\begin{aligned}\theta_f(0) = \theta_f(0 * x) &= \left(\theta_f(0) * f(x)\right) \wedge \left(f(0) * \theta_f(x)\right)\\ &= \left(\theta_f(0) * f(x)\right) \wedge (0 * \theta_f(x))\\ &= \left(\theta_f(0) * f(x)\right) \wedge 0\\ &= 0 * (0 * \left(\theta_f(0) * f(x)\right)\\ &= 0 * 0\\ &= 0.\end{aligned}$$

The following result gives a necessary and sufficient condition for the derivation θ_f to be regular.

Proposition 4.5: Let θ_f be a self map of a BH-algebra X then the following hold:

1. If θ_f is an (l, r) - f- derivation of X then $\theta_f(x) = \theta_f(x) \wedge f(x)$ for all $x \in X$, if and only if $\theta_f(0) = 0$.

2. If θ_f is an (r, l) - f- derivation of X then $\theta_f(x) = f(x) \wedge \theta_f(x)$ for all $x \in X$, if and only if $\theta_f(0) = 0$.

Proof:

1. Let θ_f be an (l, r)- f- derivation of X such that $\theta_f(0) = 0$. Then,

$$\theta_f(x) = \theta_f(x * 0)$$
$$= \left(\theta_f(x) * f(0)\right) \wedge \left(f(x) * \theta_f(0)\right)$$
$$= \left(\theta_f(x) * 0\right) \wedge \left(f(x) * \theta_f(0)\right)$$
$$= \theta_f(x) \wedge (f(x) * 0)$$
$$= \theta_f(x) \wedge f(x)$$

Conversely, assume that $\theta_f(x) = \theta_f(x) \wedge f(x)$ for all $x \in X$.

Now, $\theta_f(0) = \theta_f(0) \wedge f(0)$

$$= \theta_f(0) \wedge 0$$
$$= 0 * \left(0 * \theta_f(0)\right)$$
$$= 0 * 0$$
$$= 0$$

2. Let θ_f be an (r, l)- f- derivation of X such that $\theta_f(0) = 0$. Then,

$$\theta_f(x) = \theta_f(x * 0)$$
$$= \left(f(x) * \theta_f(0)\right) \wedge \left(\theta_f(x) * f(0)\right)$$
$$= \left(f(x) * \theta_f(0)\right) \wedge \left(\theta_f(x) * 0\right)$$
$$= (f(x) * 0) \wedge \left(\theta_f(x)\right)$$
$$= f(x) \wedge \theta_f(x)$$

Conversely, assume that $\theta_f(x) = f(x) \wedge \theta_f(x)$ for all $x \in X$.

Then $\theta_f(0) = f(0) \wedge \theta_f(0)$

$$= 0 \wedge \theta_f(0)$$
$$= \theta_f(0) * \left(\theta_f(0) * 0\right)$$
$$= \theta_f(0) * \theta_f(0)$$
$$= 0\ .$$

Remark 4.6: If θ_f is a f-derivation of a BH- algebra X, θ_f is a self map, therefore we have,

$$\big(f(x) * \big(f(x) * \theta_f(x)\big)\big) * f(x) = \big(\theta_f(x) * \big(\theta_f(x) * f(x)\big)\big) * f(x) = 0.$$

Proposition 4.7: Let θ_f be an f-derivation of a BH-algebra X. Then the following hold:

1. Ker θ_f is a subalgebra of X.

2. $\theta_f(x) \leq f(x)$, for all $x \in X$.

Proof:

1. Let $x, y \in Ker\ \theta_f$ then $\theta_f(x) = 0 = \theta_f(y)$ and

$$\begin{aligned}\theta_f(x * y) &= \big(f(x) * \theta_f(y)\big) \wedge \big(\theta_f(x) * f(y)\big)\\ &= (f(x) * 0) \wedge \big(0 * f(y)\big)\\ &= (f(x) * 0) \wedge 0\\ &= f(x) \wedge 0\\ &= 0 * \big(0 * f(x)\big)\\ &= 0\end{aligned}$$

(ie) $\theta_f(x * y) = 0 \Rightarrow x * y \in Ker\theta_f$.

Hence $x, y \in Ker\theta_f \Rightarrow x * y \in Ker\theta_f$

$Ker\theta_f$ is a subalgebra of X.

2. By Prop 4.5(1)

$$\begin{aligned}\theta_f(x) &= \theta_f(x) \wedge f(x), \forall x \in X\\ &= f(x) * \big(f(x) * \theta_f(x)\big)\end{aligned}$$

Also $\theta_f(x) * f(x) = (f(x) * (f(x) * \theta_f(x))) * f(x)$

$= 0\ (by\ remark\ 4.6)$

Then $\theta_f(x) \leq f(x)$.

Proposition 4.8: Let θ_f be a f-derivation of a BH-algebra X and $a \in X$ such that $\theta_f(x) * a = 0$ and $\theta_f(x) * f(a) = 0, \forall x \in X$. Then θ_f is a regular f-derivation on X.

Proof:

Let θ_f be a f-derivation of a BH-algebra X and $a \in X$ such that $\theta_f(x) * a = 0$ and $\theta_f(x) * f(a) = 0, \forall x \in X$.

Claim:
θ_f is a regular f- derivation on X.(ie) $\theta_f(0) = 0$.

$$\begin{aligned}\theta_f(0) &= \theta_f(0 * a)\\ &= \big(\theta_f(0) * f(a)\big) \wedge \big(f(0) * \theta_f(a)\big)\ (\because\ \theta_f\ is\ (l,r) derivation)\\ &= \big(\theta_f(0) * f(a)\big) \wedge \big(0 * \theta_f(a)\big)\end{aligned}$$

$$= \left(\theta_f(0) * f(a)\right) \wedge 0$$
$$= 0 * \left(0 * \left(\theta_f(0) * f(a)\right)\right)$$
$$= 0 * \left(\theta_f(0) * f(a)\right)$$
$$= 0 * 0$$
$$= 0$$

Hence θ_f is a regular f-derivation on X.

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