TWO TYPES OF GENERALIZED ULAM - HYERS STABILITY OF A α – TYPE CAUCHY – JENSEN FUNCTIONAL EQUATION IN INTUITIONISTIC FUZZY BANACH SPACES

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Abstract: The key intend of this paper is to scrutinize the generalized Ulam- Hyers stability of a α – type Cauchy – Jensen Functional Equation in Intuitionistic Fuzzy Banach Spaces with the help of Hyers Type and Fixed Point Type.

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1. Introduction

In [29], Ulam proposed the general Ulam stability problem: When is it true that by slightly changing the hypotheses of a theorem one can still assert that the thesis of the theorem remains true or approximately true? In [14], Hyers gave the former assenting answer to the question of Ulam for additive functional equations on Banach spaces. Hyers result has since then seen countless considerable generalizations, both in terms of the control condition used to define the concept of approximate solution [3, 12, 23, 24, 25].

The solution and stability of different functional equations in various normed spaces were introduced and discussed in [1,2,4,5,6,8,9,15,16,17,22] and reference cited there in.

Very recently, Iz-iddine EL-Fassi, Samir Kabbaj [11], introduced and investigate the generalized Hyers- Ulam stability of homomorphisms in Banach algebras of a α – Cauchy-Jensen type functional equation

$$f\left(\frac{x+y}{\alpha}+z\right)+f\left(\frac{x-y}{\alpha}+z\right)=\frac{2}{\alpha}f(x)+2f(z) \tag{1.1}$$

where $\alpha \in \mathbb{N}_{\geq 2}$.

2. Preliminaries On Intuitionistic Fuzzy Banach Spaces

In this section, using the idea of Intuitionistic fuzzy metric spaces introduced by J.H. Park [21] and R. Saadati and J.H. Park [26,27], we define the new notion of intuitionistic fuzzy Banach spaces with the help of the notion of continuous t – representable (see [30]).

Lemma 2.1 [15] Consider the set L^* and the order relation $\leq_{L^*}$ defined by:

$$L^* = \{(x_1, x_2) : (x_1, x_2) \in [0,1]^2 \; and \; x_1 + x_2 \leq 1\},$$

$(x_1,x_2) \leq_{L^*} (y_1,y_2) \Leftrightarrow x_1 \leq y_1, x_2 \geq y_2, \; \forall (x_1,x_2),(y_1,y_2) \in L^*$ Then $(L^*, \leq_{L^*})$ is a complete lattice.

Definition 2.2 [12] An intuitionistic fuzzy set $A_{\zeta,\eta}$ in a universal set U is an object

$$A_{\zeta,\eta} = \{(\zeta_A(u), \eta_A(u)) | u \in U\}$$

for all $u \in U$, $\zeta_A(u) \in [0,1]$ and $\eta_A(u) \in [0,1]$ are called the membership degree and the non-membership degree, respectively, of u in $A_{\zeta,\eta}$ and, furthermore, they satisfy $\zeta_A(u) + \eta_A(u) \leq 1$.

We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. Classically, a triangular norm $* = T$ on $[0,1]$ is defined as an increasing, commutative, associative mapping $T : [0,1]^2 \to [0,1]$ satisfying $T(1, x) = 1 * x = x$ for all $x \in [0,1]$. A triangular conorm $S = \blacklozenge$ is defined as an increasing, commutative, associative mapping $S : [0,1]^2 \to [0,1]$ satisfying $S(0, x) = 0 \blacklozenge x = x$ for all $x \in [0,1]$.

Using the lattice $(L^*, \le_{L^*})$, these definitions can be straightforwardly extended.

Definition 2.3 [12] A triangular norm ($t-$norm) on L^* is a mapping $T:(L^*)^2 \to L^*$ satisfying the following conditions:

- $(\forall \in L^*)(T(x,1_{L^*}) = x)$ (boundary condition);
- $(\forall (x,y) \in (L^*)^2)(T(x,y) = T(y,x))$ (commutativity);
- $(\forall (x,y,z) \in (L^*)^3)(T(x,T(y,z)) = T(T(x,y),z))$ (associativity);
- $(\forall (x,x',y,y') \in (L^*)^4)(x \le_{L^*} x' \text{ and } y \le_{L^*} y' \Rightarrow T(x,y) \le_{L^*} T(x',y'))$ (monotonicity).

If $(L^*, \le_{L^*}, T)$ is an Abelian topological monoid with unit 1_{L^*}, then L^* is said to be a continuous $t-$norm.

Definition 2.4. [12] A continuous t-norms T on L* is said to be continuous t-representable if there exist a continuous t-norm $*$ and a continuous $t-$conorm $\blacklozenge$ on $[0,1]$ such that, for all $x=(x_1,x_2), y=(y_1,y_2) \in L^*$, $T(x,y) = (x_1 * y_1, x_2 \blacklozenge y_2)$.

For example,

$T(a,b) = (a_1b_1, \min\{a_2+b_2,1\})$ *and* $M(a,b) = (\min\{a_1,b_1\}, \max\{a_2,b_2\})$ for all $a=(a_1,a_2), b=(b_1,b_2) \in L^*$ are continuous $t-$representable. Now, we define a sequence T^n recursively by $T^1 = T$ and

$$T^n\left(x^{(1)},\ldots,x^{(n+1)}\right) = T\left(T^{n-1}\left(x^{(1)},\ldots,x^{(n)}\right), x^{(n+1)}\right), \quad \forall n \ge 2, x^{(i)} \in L^*.$$

Definition 2.5 [32] A negator on L^* is any decreasing mapping $N: L^* \to L^*$ satisfying $N:(0_{L^*}) = 1_{L^*}$ and $N(1_{L^*}) = 0_{L^*}$. If $N(N(x)) = x$ for all $x \in L^*$, then N is called an involutive negator. A negator on $[0,1]$ is a decreasing mapping $N:[0,1] \to [0,1]$ satisfying $N(0)=1$ and $N(1)=0$. N_s denotes the standard negator on $[0,1]$ defined by $N_s(x) = 1-x, \ \forall x \in [0,1]$.

Definition 2.6 [32] Let μ and ν be membership and nonmembership degree of an intuitionistic fuzzy set from $X \times (0,+\infty)$ to $[0,1]$ such that $\mu_x(t) + \nu_x(t) \le 1$ for all $x \in X$ and all $t>0$. The triple $(X, P_{\mu,\nu}, T)$ is said to be an intuitionistic fuzzy normed space (briefly IFN-space) if X is a vector space, T is a continuous $t-$representable and $P_{\mu,\nu}$ is a mapping $X \times (0,+\infty) \to L^*$ satisfying the following conditions: for all $x,y \in X$ and $t,s>0$,

(IFNS1) $P_{\mu,\nu}(x,0) = 0_{L^*}$;

(IFNS2) $P_{\mu,\nu}(x,t) = 1_{L^*}$ if and only if $x=0$;

(IFNS3) $P_{\mu,\nu}(\delta x,t) = P_{\mu,\nu}\left(x, \frac{t}{|\delta|}\right)$ for all $\delta \ne 0$;

(IFNS4) $P_{\mu,\nu}(x+y,t+s) \ge_{L^*} T(P_{\mu,\nu}(x,t), P_{\mu,\nu}(y,s))$.

In this case, $P_{\mu,\nu}$ is called an intuitionistic fuzzy norm. Here, $P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t))$.

Example 2.7 [32] Let $(X, \|\cdot\|)$ be a normed space. Let $T(a,b) = (a,b,\min(a_2+b_2,1))$ for all $a=(a_1,a_2), b=(b_1,b_2) \in L^*$ and μ,ν be membership and non-membership degree of an intuitionistic fuzzy set defined by

$$P_{\mu,\nu}(x,t) = (\mu_x(t), \nu_x(t)) = \left(\frac{t}{t+\|x\|}, \frac{\|x\|}{t+\|x\|}\right), \forall \ \ t \in R^+.$$

Then $(X, P_{\mu,\nu}, T)$ is an IFN-sapce.

Definition 2.8 [32] A sequence $\{x_n\}$ in an IFN-space $(X, P_{\mu,\nu}, T)$ is called a Cauchy sequence if, for any $\varepsilon > 0$ and $t>0$, there exists $n_0 \in N$ such that

$$P_{\mu,\nu}(x_n - x_m, t) > L^*(N_s(\varepsilon), \varepsilon), \ \forall \ \ n,m \ge n_0,$$

where N_s is the standard negator.

Definition 2.9 [32] The sequence $\{x_n\}$ is said to be convergent to a point $x \in X$ (denoted by $x_n \overset{P_{\mu,\nu}}{\rightarrow} x$ if $P_{\mu,\nu}(x_n - x, t) \rightarrow 1_{L^*}$ as $n \rightarrow \infty$ for every $t > 0$.

Definition 2.10 [32] An IFN-space $(X, P_{\mu,\nu}, T)$ is said to be complete if every Cauchy sequence in X is convergent to a point $x \in X$.

In order to establish the stability results let us consider the following:

- X - Linear space.
- $(Z, P'_{\mu,\nu}, M)$ - IFN-space.
- $(Y, P'_{\mu,\nu}, M)$ - Complete IFN-space (IFBS).
- $\gamma \in \{1, -1\}$.
- Define a constant θ_i by

$$\theta_i = \begin{cases} \beta & if \quad i = 0; \\ \frac{1}{\beta} & if \quad i = 1. \end{cases} \quad \text{where } \beta = \left(\frac{2}{\alpha} + 1\right). \tag{2.1}$$

3. Intuitionistic Fuzzy Banach Space: Stability Results: Hyers Type

In this section, we check the generalized Ulam – Hyers stability of the Cauchy-Jensen functional equation (1.1) in intuitionistic fuzzy Banach space with the help of Hyers type.

Theorem 3.1 Let $\omega : X^3 \rightarrow Z$ be a function such that for some $0 < \left(\frac{a}{\beta}\right)^{\gamma} < 1$,

$$P'_{\mu,\nu}\left(\omega\left(\beta^{q\gamma}x, \beta^{q\gamma}x, \beta^{q\gamma}x\right), r\right) \geq_{L^*} P'_{\mu,\nu}\left(a^{q\gamma}\, \omega(x,x,x), r\right) \tag{3.1}$$

for all $x \in X$ and all $r > 0$ and

$$\lim_{q\to\infty} P'_{\mu,\nu}\left(\omega\left(\beta^{q\gamma}x, \beta^{q\gamma}y, \beta^{q\gamma}z\right), \beta^{q\gamma}r\right) = 1_{L^*} \tag{3.2}$$

for all $x, y, z \in X$ and $r > 0$. If $f : X \rightarrow Y$ is a mapping such that

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{\alpha} + z\right) + f\left(\frac{x-y}{\alpha} + z\right) - \frac{2}{\alpha} f(x) - 2f(z), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\omega(x,.y,z), r\right) \tag{3.3}$$

for all $x, y, z \in X$ and $r > 0$. Then the limit

$$P_{\mu,\nu}\left(A(x) - \frac{f(\beta^{q\gamma}x)}{\beta^{q\gamma}}, r\right) \rightarrow 1^*_L, \qquad as \quad q \quad \rightarrow \infty, \quad r > 0 \tag{3.4}$$

exists for all $x \in X$ and the mapping $A : X \rightarrow Y$ is the unique additive mapping such that

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\omega(x,x,x), \frac{|\beta - a|\, r}{\beta}\right) \tag{3.5}$$

for all $x \in X$ and all $r > 0$.

Proof. Case 1 : $\gamma = 1$. Changing (x, y, z) by (x, x, x) in (3.3), we get

$$P_{\mu,\nu}\left(f\left(\left(\frac{2}{\alpha}+1\right)x\right) - \left(\frac{2}{\alpha}+1\right) f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\omega(x,x,x), r\right) \Rightarrow P_{\mu,\nu}\left(f(\beta x) - \beta f(x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\omega(x,x,x), r\right) \tag{3.6}$$

for all $x \in X$ and $r > 0$. Replacing x by $\beta^q x$ in (3.6) and using (3.1), (IFNS3), we arrive

$$P_{\mu,\nu}\left(\frac{f(\beta^{q+1}x)}{\beta} - f(\beta^q x), r\right) \geq_{L^*} P'_{\mu,\nu}\left(\omega(x,x,x), \frac{r}{a^q}\right) \tag{3.7}$$

for all $x \in X$ and $r > 0$. It is easy to verify from (3.7) and replacing r by $a^q r$ in the resulting inequality, we have

$$P_{\mu,\nu}\left(\frac{f(\beta^q x)}{\beta^q}-\frac{f(\beta^{q+1}x)}{\beta^{q+1}},\left[\frac{a}{\beta}\right]^q r\right)\geq_{L^*} P'_{\mu,\nu}\left(\omega\left(x,x,x\right),r\right) \tag{3.8}$$

for all $x\in X$ and all $r>0$. It is easy to see that

$$\frac{f(\beta^q x)}{\beta^q}-f(x)=\sum_{j=0}^{q-1}\frac{f(\beta^{j+1}x)}{\beta^{(j+1)}}-\frac{f(\beta^j x)}{\beta^j} \tag{3.9}$$

for all $x\in X$. From equations (3.8) and (3.9), we get

$$P_{\mu,\nu}\left(\frac{f(\beta^q x)}{\beta^q}-f(x),\sum_{j=0}^{q-1}\left[\frac{a}{\beta}\right]^j\cdot r\right)\geq_{L^*} M_{j=0}^{q-1}P'_{\mu,\nu}\left(\sum_{j=0}^{q-1}\frac{f(\beta^{j+1}x)}{\beta^{(j+1)}}-\frac{f(\beta^j x)}{\beta^j},\sum_{j=0}^{q-1}\left[\frac{a}{\beta}\right]^j\cdot r\right)$$

$$\geq_{L^*} M_{j=0}^{q-1}P'_{\mu,\nu}\left(\omega\left(x,x,x\right),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\omega\left(x,x,x\right),r\right) \tag{3.10}$$

for all $x\in X$ and all $r>0$. Replacing x by $\beta^p x$ in (3.10), using (3.1), (IFNS3) and replacing r by $a^p r$ and one again using (IFNS3), we obtain

$$P_{\mu,\nu}\left(\frac{f(\beta^{q+p}x)}{\beta^{q+p}}-\frac{f(\beta^p x)}{\beta^p},r\right)\geq_{L^*} P'_{\mu,\nu}\left(\omega(x,x,x),r\Big/\sum_{j=p}^{q-1}\left[\frac{a}{\beta}\right]^j\right) \tag{3.11}$$

for all $x\in X$ and all $r>0$ and all $p,q\geq 0$.

By definition of convergence, we see the sequence $\left\{\frac{f(\beta^q x)}{\beta^q}\right\}$ is a Cauchy sequence in $\left(Y,P_{\mu,\nu},M\right)$. Since $\left(Y,P_{\mu,\nu},M\right)$ is a complete IFN-space this sequence convergent to some point $A(x)\in Y$. So one can define the mapping $A:X\to Y$ by $P_{\mu,\nu}\left(A(x)-\frac{f(\beta^q x)}{\beta^q},r\right)\to 1_L^*, as\ q\to\infty, r>0$ for all $x\in X$. Letting $p=0$ and letting limit as $q\to\infty$ in (3.11), we get

$$P_{\mu,\nu}\left(A(x)-f(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\omega\left(x,x,x\right),\frac{(\beta-a)\,r}{\beta}\right) \tag{3.12}$$

for all $x\in X$ and all $r>0$. To prove A satisfies the (1.1), replacing (x,y,z) by $(\beta^q x,\beta^q y,\beta^q z)$ in (3.3), we have

$$P_{\mu,\nu}\left(\frac{1}{\beta^q}\left[f\left(\frac{\beta^q\left(x+y\right)}{\alpha}+\beta^q z\right)+f\left(\frac{\beta^q\left(x-y\right)}{\alpha}+\beta^q z\right)-\frac{2}{\alpha}f\left(\beta^q x\right)-2f\left(\beta^q z\right)\right],r\right)\geq_{L^*} P'_{\mu,\nu}\left(\omega\left(\beta^q x,\beta^q y,\beta^q z\right),\beta^q r\right) \tag{3.13}$$

for all $x,y,z\in X$ and $r>0$. Now,

$$\begin{aligned}
&P_{\mu,\nu}\left(A\left(\frac{x+y}{\alpha}+z\right)+A\left(\frac{x-y}{\alpha}+z\right)-\frac{2}{\alpha}A\left(x\right)-2A\left(z\right),r\right)\\
&\geq_{L^*} M^2\left\{P_{\mu,\nu}\left(A\left(\frac{x+y}{\alpha}+z\right)-\frac{1}{\beta^q}f\left(\frac{\beta^q\left(x+y\right)}{\alpha}+\beta^q z\right),\frac{r}{5}\right),P_{\mu,\nu}\left(A\left(\frac{x-y}{\alpha}+z\right)-\frac{1}{\beta^q}f\left(\frac{\beta^q\left(x-y\right)}{\alpha}+\beta^q z\right),\frac{r}{5}\right),\right.\\
&\quad P_{\mu,\nu}\left(-\frac{2}{\alpha}A\left(x\right)+\frac{1}{\beta^q}\frac{2}{\alpha}f\left(\beta^q x\right),\frac{r}{5}\right),P_{\mu,\nu}\left(-2A\left(z\right)+\frac{1}{\beta^q}2f\left(\beta^q z\right),\frac{r}{5}\right),\\
&\quad\left.P_{\mu,\nu}\left(\frac{1}{\beta^q}\left[f\left(\frac{\beta^q\left(x+y\right)}{\alpha}+\beta^q z\right)+f\left(\frac{\beta^q\left(x-y\right)}{\alpha}+\beta^q z\right)-\frac{2}{\alpha}f\left(\beta^q x\right)-2f\left(\beta^q z\right)\right],\frac{r}{5}\right)\right\}
\end{aligned} \tag{3.14}$$

for all $x,y,z\in X$ and all $r>0$. Letting $q\to\infty$ in (3.14) and using (3.13), (3.2), and (IFNS2), we see that A satisfies the functional equation (1.1). In order to prove $A(x)$ is unique, let $A'(x)$ be another additive functional equation satisfying (1.1) and (3.5). Hence,

$$P_{\mu,\nu}(A(x)-A'(x),r)=P_{\mu,\nu}\left(\frac{A(\beta^q x)}{\beta^q}-\frac{A'(\beta^q x)}{\beta^q},r\right)\geq_{L^*} M\left(P_{\mu,\nu}\left(\frac{A(\beta^q x)}{\beta^q}-\frac{f(\beta^q x)}{\beta^q},\frac{r}{2}\right),P_{\mu,\nu}\left(\frac{f(\beta^q x)}{\beta^q}-\frac{A'(\beta^q x)}{\beta^q},\frac{r}{2}\right)\right)$$
$$\geq_{L^*} P'_{\mu,\nu}\left(\omega(\beta^q x,\beta^q x\beta^q x),\frac{r(\beta-a)}{2\beta}\right)=P'_{\mu,\nu}\left(\omega(x,x,x),\frac{r(\beta-a)}{2\beta a^q}\right)$$

for all $x\in X$ and all $r>0$. Since $\lim_{q\to\infty}P'_{\mu,\nu}\left(\omega(x,x,x),\frac{r(\beta-a)}{2\beta a^q}\right)=1_{L^*}$. Thus $P_{\mu,\nu}(A(x)-A'(x),r)=1_{L^*}$ for all $x\in X$ and all $r>0$, hence $A(x)=A'(x)$. Therefore $A(x)$ is unique. Thus, the theorem holds for $\gamma=1$. For $\gamma=-1$, we can prove the result by a similar method. This completes the proof of the theorem.

The subsequent corollaries are immediate consequence of Theorem 3.1 concerning the stabilities of (1.1).

Corollary 3.2 If $f:X\to Y$ is a mapping satisfying the functional inequality

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{\alpha}+z\right)+f\left(\frac{x-y}{\alpha}+z\right)-\frac{2}{\alpha}f(x)-2f(z),r\right)\geq_{L^*} P'_{\mu,\nu}(\rho,r) \tag{3.15}$$

for all $x,y,z\in X$ and all $r>0$ and $\rho>0$ be a constant, then there exists a unique additive mapping $A:X\to Y$ such that

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\rho,\frac{|\beta-1|r}{\beta}\right) \tag{3.16}$$

for all $x\in X$ and $r>0$.

Corollary 3.3 If $f:X\to Y$ is a mapping satisfying the functional inequality

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{\alpha}+z\right)+f\left(\frac{x-y}{\alpha}+z\right)-\frac{2}{\alpha}f(x)-2f(z),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\rho\left(|x|^t+|y|^t+|z|^t\right),r\right) \tag{3.17}$$

for all $x,y,z\in X$ and all $r>0$ and ρ,t are positive constants with $t\neq 1$, then there exists a unique additive mapping $A:X\to Y$ such that

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\rho,\frac{|\beta-\beta^t|r}{\beta}\right) \tag{3.18}$$

for all $x\in X$ and $r>0$.

Corollary 3.3 If $f:X\to Y$ is a mapping satisfying the functional inequality

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{\alpha}+z\right)+f\left(\frac{x-y}{\alpha}+z\right)-\frac{2}{\alpha}f(x)-2f(z),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\rho\left(|x|^t|y|^t|z|^t+\left\{|x|^t+|y|^t+|z|^t\right\}\right),r\right) \tag{3.19}$$

for all $x,y,z\in X$ and all $r>0$ and ρ,t are positive constants with $3t\neq 1$, then there exists a unique additive mapping $A:X\to Y$ such that

$$P_{\mu,\nu}\left(f(x)-A(x),r\right)\geq_{L^*} P'_{\mu,\nu}\left(\rho,\frac{|\beta-\beta^{3t}|r}{\beta}\right) \tag{3.20}$$

for all $x\in X$ and $r>0$.

4. Fixed Point Type

Now, we present the following theorem due to B. Margolis and J.B. Diaz [18] for the fixed point theory.

Theorem 4.1 [18] Suppose that for a complete generalized metric space (Ω,d) and a strictly contractive mapping $T:\Omega\to\Omega$ with Lipschitz constant L. Then, for each given $x\in\Omega$, either

$$d(T^n x,T^{n+1}x)=\infty,\qquad \forall \quad n\geq 0,$$

or there exists a natural number n_0 such that the properties hold:

(FP1) $d(T^n x,T^{n+1}x)<\infty$ for all $n\geq n_0$;

(FP2) The sequence $(T^n x)$ is convergent to a fixed to a fixed point y^* of T;

(FP3) y^* is the unique fixed point of T in the set $\Delta = \{y \in \Omega : d(T^{n_0}x, y) < \infty\}$;

(FP4) $d(y^*, y) \le \frac{L^{i-1}}{1-L} d(y, Ty)$ for all $y \in \Delta$.

Theorem 4.2 Assume $\omega : X^N \to Z$ be a function satisfying the condition

$$\lim_{q\to\infty} P'_{\mu,\nu}\left(\omega\left(\theta_i^q x, \theta_i^q y, \theta_i^q z\right), \theta_i^q r\right) = 1_{L^*} \tag{4.1}$$

for all $x, y, z \in X$ and $r > 0$. If $f : X \to Y$ is a mapping such that

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{\alpha} + z\right) + f\left(\frac{x-y}{\alpha} + z\right) - \frac{2}{\alpha} f(x) - 2f(z), r\right) \ge_{L^*} P'_{\mu,\nu}\left(\omega(x, y, z), r\right) \tag{4.2}$$

for all $x, y, z \in X$ and $r > 0$. If there exist a constant $L = L(i)$ such that

$$\omega(x, x, x) = \beta\, \omega\left(\frac{x}{\beta}, \frac{x}{\beta}, \frac{x}{\beta}\right) \tag{4.3}$$

with the property

$$P'_{\mu,\nu}\left(\frac{1}{\theta_i} \omega\left(\theta_i x, \theta_i x, \theta_i x\right), r\right) = P'_{\mu,\nu}\left(L\, \omega(x, x, x), r\right) \tag{4.4}$$

for all $x \in X$ and all $r > 0$. Then there exists a the mapping $A : X \to Y$ which is a unique additive mapping such that

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \ge_{L^*} P'_{\mu,\nu}\left(\frac{L^{1-i}}{1-L} \omega(x, x, x), r\right) \tag{4.5}$$

for all $x \in X$ and all $r > 0$.

Proof. Consider the set $\Psi = \{g_1 / g_1 : X \to Y, g_1(0) = 0\}$ and introduce the generalized metric on Ψ

$$d(g_1, g_2) = \inf\{\eta \in (0, \infty) : P_{\mu,\nu}\left(g_1(x) - g_2(x), r\right) \ge_{L^*} P'_{\mu,\nu}\left(\eta\, \omega(x, x, x), r\right) \tag{4.6}$$

for all $x \in X$ and all $r > 0$. It is easy to see that (4.6) is complete with respect to the defined metric. Define $J : \Psi \to \Psi$ by $J\, g_1(x) = \frac{1}{\theta_i} g_1(\theta_i x)$, for all $x \in X$. Now, from (4.6) and $g_1, g_2 \in \Psi$, we arrive J is a strictly contractive mapping on Ψ with Lipschitz constant L (see [22]). Changing (x, y, z) by (x, x, x) in (4.2) with the help of (4.3), (4.4) and for the case $i = 0$, we reach

$$P_{\mu,\nu}\left(f(x) - Jf(x), r\right) \ge_{L^*} P'_{\mu,\nu}\left(L\omega(x, x, x), r\right) \tag{4.7}$$

for all $x \in X$ and $r > 0$. Again changing (x, y, z) by $\left(\frac{x}{\beta}, \frac{x}{\beta}, \frac{x}{\beta}\right)$ in (4.2) with the help of (4.3), (4.4) and for the case $i = 1$, we again reach

$$P_{\mu,\nu}\left(f(x) - Jf(x), r\right) \ge_{L^*} P'_{\mu,\nu}\left(\omega(x, x, x), r\right) \tag{4.8}$$

for all $x \in X$ and $r > 0$. Thus, from (4.7) and (4.8), we arrive

$$P_{\mu,\nu}\left(f(x) - Jf(x), r\right) \ge_{L^*} P'_{\mu,\nu}\left(L^{i-1}\omega(x, x, x), r\right) \tag{4.9}$$

for all $x \in X$ and $r > 0$. Hence property (FP1) of Theorem 4.1 holds. It follows from property (FP2) of Theorem 4.1 that there exists a fixed point A of J in Ψ such that $\frac{f(\theta_i^q x)}{\theta_i^q} \xrightarrow{P_{\mu,\nu}} A(x)$ *as* $q \to \infty$ for all $x \in X$ and $r > 0$. In order to show that A satisfies (1.1), the proof is similar lines to tat of Thorem 3.1 By property (FP3) of Theorem 4.1, A is the unique fixed point of J in the set $\Delta = \{A \in \Psi : d(f, A) < \infty\}$, such that

$$P_{\mu,\nu}\left(f(x) - A(x), r\right) \ge_{L^*} P'_{\mu,\nu}\left(\eta\, \omega(x, x, x), r\right)$$

for all $x \in X$ and $r > 0$. Finally by property (FP4) of Theorem 4.1 , we obtain our desired inequality (4.5). This completes the proof of the theorem.

The following corollary is an immediate consequence of Theorem 3.4 concerning the stability of (1.1).

Corollary 3.5 If $f : X \to Y$ is a mapping satisfying the functional inequality

$$P_{\mu,\nu}\left(f\left(\frac{x+y}{\alpha}+z\right)+f\left(\frac{x-y}{\alpha}+z\right)-\frac{2}{\alpha}f(x)-2f(z),r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}(\rho,r) \\ P'_{\mu,\nu}\left(\rho\left(|x|^t+|y|^t+|z|^t\right),r\right) \\ P'_{\mu,\nu}\left(\rho\left(|x|^t|y|^t|z|^t+\left\{|x|^t+|y|^t+|z|^t\right\}\right),r\right) \end{cases} \tag{4.10}$$

for all $x, y, z \in X$ and all $r > 0$ and ρ, t are positive constants, then there exists a unique additive mapping $A : X \to Y$ such that

$$P_{\mu,\nu}\left(f(x)-A(x),r\right) \geq_{L^*} \begin{cases} P'_{\mu,\nu}\left(\beta\rho,|\beta-1|r\right); \\ P'_{\mu,\nu}\left(\beta\rho|x|^t,|\beta-\beta^t|r\right); \\ P'_{\mu,\nu}\left(\beta\rho,|\beta-\beta^{3t}|r\right) \end{cases} \tag{4.11}$$

for all $x \in X$ and $r > 0$.

Proof. Setting $P'_{\mu,\nu}(\omega(x,y,z),r) = \begin{cases} P'_{\mu,\nu}(\rho,r) \\ P'_{\mu,\nu}\left(\rho\left(|x|^t+|y|^t+|z|^t\right),r\right) \\ P'_{\mu,\nu}\left(\rho\left(|x|^t|y|^t|z|^t+\left\{|x|^t+|y|^t+|z|^t\right\}\right),r\right) \end{cases}$ for all $x, y, z \in X$.

Then one can easy to verify that (4.1) holds. It follows from (4.3), (4.4) and above setting , we have

$$P'_{\mu,\nu}\left(\omega(x,x,x),r\right) = P'_{\mu,\nu}\left(\beta\,\omega\left(\frac{x}{\beta},\frac{x}{\beta},\frac{x}{\beta}\right),r\right) = \begin{cases} P'_{\mu,\nu}\left(\beta\rho,r\right); \\ P'_{\mu,\nu}\left(3\beta^{i-t}\rho\|^t,r\right); \\ P'_{\mu,\nu}\left(\beta^{1-3t}\rho|x|^{3t},r\right); \end{cases} \tag{4.12}$$

and

$$P'_{\mu,\nu}\left(\frac{1}{\theta_i}\omega(\theta_i x,\theta_i x,\theta_i x),r\right) = \begin{cases} P'_{\mu,\nu}\left(\theta_i^{-1}N\rho,r\right); \\ P'_{\mu,\nu}\left(\theta_i^{t-1}N\rho|x|^t,r\right); \\ P'_{\mu,\nu}\left(\theta_i^{3t-1}N\rho|x|^t,r\right); \end{cases} = P'_{\mu,\nu}\left(L\,\omega(x,x,x),r\right) \tag{4.13}$$

for all $x \in X$. Hence, in view of (4.5) and (4.13) , we arrive our inequality (4.11). Hence the proof is complete.

REFERENCES

[1]. **Abasalt Bodaghi, Choonkil Park, John Michael Rassias**, Fundamental stabilities of the nonic functional equation in intuitionistic fuzzy normed spaces, Commun. Korean Math. Soc. 31 (2016), No. 4, 729-743.

[2]. **J. Aczel and J. Dhombres**, Functional Equations in Several Variables, Cambridge Univ, Press, 1989.

[3]. **T. Aoki**, On the stability of the linear transformation in Banach spaces, J. Math. Soc. Japan, 2 (1950), 64-66.

[4]. **M. Arunkumar, G. Ganapathy, S. Murthy, S. Karthikeyan,** Stability of the Generalized Arun-additive functional equation in Intuitionistic fuzzy normed spaces, International Journal Mathematical Sciences and Engineering Applications, Vol.4, No. V, (2010), 135-146.

[5]. **M. Arunkumar, S. Karthikeyan,** Solution and Intuitionistic Fuzzy stability of n- dimensional quadratic functional equation: Direct and Fixed Point Methods, International Journal of Advanced Mathematical Sciences (IJAMS), Vol 2 (1), (2014), 21-33.

[6]. **M. Arunkumar, John M Rassias, S. Karthikeyan,** Stability Of A Leibniz Type Additive And Quadratic Functional Equation In Intuitionistic Fuzzy Normed Spaces, Advances in Theoretical and Applied Mathematics, Volume 11, Number 2 (2016), pp. 145-169.

[7]. **K.T. Atanassov**, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20 (1986), 87-96.

[8]. **L. Cadariu, V. Radu**, On the stability of the Cauchy functional equation: A fixed point approach, Grazer Math. Ber. 346 (2004), 43-52.

[9]. **S. Czerwik**, Functional Equations and Inequalities in Several Variables, World Scientific, River Edge, NJ, 2002.

[10]. **G. Deschrijver, E.E. Kerre** On the relationship between some extensions of fuzzy set theory, Fuzzy Sets and Systems 23 (2003), 227-235.

[11]. **I.EL. Fassi, S. Kabbaj,** On the Stability of α Cauchy-Jensen type functional equation in Banach Algebras, Malaya J. Mat. 4(1)(2016) 169–177.

[12]. **P. Gavruta**, A generalization of the Hyers-Ulam-Rassias stability of approximately additive mappings , J. Math. Anal. Appl., 184 (1994), 431-436.

[13]. **S.B. Hosseini, D. O'Regan, R. Saadati**, Some results on intuitionistic fuzzy spaces, Iranian J. Fuzzy Syst, 4 (2007) 53- 64.

[14]. **D.H. Hyers**, On the stability of the linear functional equation, Proc.Nat. Acad.Sci.,U.S.A.,27 (1941) 222-224.

[15]. **D.H. Hyers, G. Isac, Th.M. Rassias**, Stability of functional equations in several variables, Birkhauser, Basel, 1998.

[16]. **S.M. Jung**, Hyers-Ulam-Rassias Stability of Functional Equations in Mathematical Analysis, Hadronic Press, Palm Harbor, 2001.

[17]. **Pl. Kannappan**, Functional Equations and Inequalities with Applications, Springer Monographs in Mathematics, 2009.

[18]. **B. Margoils and J. B. Diaz**, A fixed point theorem of the alternative for contractions on a generalized complete metric space, Bull. Amer. Math. Soc., 126(74) (1968), 305-309.

[19]. **A.K. Mirmostafaee, M.S. Moslehian**, Fuzzy versions of Hyers-Ulam-Rassias theorem, Fuzzy Sets and Systems, Vol. 159, no. 6, (2008), 720 - 729.

[20]. **M. Mursaleen, and K.J. Ansari**, Stability Results in Intuitionistic Fuzzy Normed Spaces for a Cubic Functional Equation, Appl. Math. Inf. Sci. 7, No. 5, (2013), 1677-1684.

[21]. **J.H. Park**, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fractals, 22 (2004), 1039-1046.

[22]. **V. Radu**, The fixed point alternative and the stability of functional equations, in: Seminar on Fixed Point Theory Cluj- Napoca, vol. IV, 2003, in press.

[23]. **J.M. Rassias**, On approximately of approximately linear mappings by linear mappings, J. Funct. Anal. USA, 46, (1982) 126-130.

[24]. **Th.M. Rassias**, On the stability of the linear mapping in Banach spaces, Proc.Amer.Math. Soc., 72 (1978), 297-300.

[25]. **K. Ravi, M. Arunkumar and J.M. Rassias**, On the Ulam stability for the orthogonally general Euler-Lagrange type functional equation, International Journal of Mathematical Sciences, Autumn 2008 Vol.3, No. 08, 36-47.

[26]. **R. Saadati, J.H. Park**, On the intuitionistic fuzzy topological spaces, Chaos, Solitons and Fractals 27 (2006),

331-344.
[27]. **R. Saadati, J.H. Park**, Intuitionstic fuzzy Euclidean normed spaces, Commun. Math. Anal., 1 (2006), 85-90.
[28]. **S.Shakeri**, Intuitionstic fuzzy stability of Jensen type mapping, J. Nonlinear Sci. Appli. Vol.2 No. 2 (2009), 105-112.
[29]. **S.M. Ulam**, Problems in Modern Mathematics, Science Editions, Wiley, NewYork, 1964.
[30]. **Ding Xuan Zhou**, On a conjecture of Z. Ditzian, J. Approx. Theory 69 (1992), 167-172.