# A Note on Prime Numbers

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**Abstract**

Leonhard Euler was the most eminent Swiss Mathematician of 18th century who contributed the theory of Prime Numbers. The Fundamental Theorem of Arithmetic shows the importance of prime numbers. The set of Prime numbers is infinite. There are various types of prime numbers i.e. Fibonacci Prime, Lucas Prime, Ramanujan Prime and so on. In this note we try to find the connection between them and various properties of Prime numbers.

***Key words : Fibonacci Prime , Lucas Prime , Ramanujan Prime***

## Introduction

A Prime Number is a Natural Number that cannot be formed by multiplying two smaller natural numbers. It is greater than one and is divisible by one and itself. The Fundamental Theorem of Arithmetic states that every Natural number greater than one is either a prime itself or can be factorised as a product of primes that is unique up to their order. The property of being prime is called Primality .Euclid's Theorem states that there are infinitely many primes, The first 25 Primes are 2,3,5, 7, 11, 13, 17, 19, 23,29,31, 37, 41,43, 47,53,59,61,67,71,73,79,83,89,97,…..

A Fibonacci sequence is defined as $F_n$, $n \geq 1 F_0 = 0$ and $F_1=1$ , $F_n = F_{n-1} + F_{n-2}$ , while Lucas sequence is defined as $L_0 = 2$ , $L_1 = 1$, $n \geq 1$, and $L_n = L_{n-1} + L_{n-2}$

**Definition 1.1** A Fibonacci Prime is a Fibonacci Number which is prime First few Fibonacci primes are 2,3,5,13,89,233,1597,28657,514229,..It is not known that Fibonacci primes are infinite..

**Definition 1.2** Similarly Lucas Prime Number is a Lucas Number which is prime. The first few prime Lucas numbers are 2,3,7,11,29, 47,199,521,2207,3571,….

It was observed by Zerger that the product $F_6F_7F_8F_9$ is the product of the first seven prime numbers: $F_6F_7F_8F_9 = 13.21.34.55 = 510$ and $510 = 2.3.5.7.11.13.17$.

**Definition 1.3** A Pell sequence is defined as $P_n$ , $n \geq 0$ and $P_n = 2P_{n-1} + P_{n-2}$ , $P_0 = 0$ and $P_1 = 1$

**Definition 1.4** Pell Prime Number is a Pell Number which is prime . The first few Pell Primes are 5,29, 169, 5741, …

**Definition 1.5** Ramanujan Prime For $n \geq 1$ , the nth Ramanujan Prime is the smallest positive integer $R_n$ for which there are n primes between $\pi(x) - \pi(x/2) \geq n$ for all $x \geq Rn$ (OEIS, A104272 ) where $\pi(x)$ is the prime counting function.

The first few values of $\pi(n)$ for n = { 1,2,3,4,5,….n} are 0,1,2,2,3 ,3,4 : 4,4,4 ,5,5,6

For example , at n =12 there are 5 primes { 2,3,5,7,11}

In other words there are at least n primes between x/2 and x .whenever $x \geq R_n$ . The smallest such number $R_n$ , must be prime, since the function $\pi(x) - \pi(x/2)$ increase only at a prime. Equivalently

$R_n = 1 + \max \{ k: \pi(k) - \pi(k/2) = n-1 \}$

The first few Ramanujan Primes are 2,11, 17, 29 , 41,47,59,67,71,97,101,107,127,149,151,167,179,181,227,229,233,239,241,263,269,281,307,311,347,..

These Prime Numbers are used in the field of Cryptography and computer security.

**Consecutive Ramanujan Primes** : We have $R_n > P_n$ for n > 1 and $R_n \sim P_{2n}$ , as $n \rightarrow \infty$ , Thus roughly speaking , the probability of a randomly chosen prime being Ramanujan is slightly less than 1/2

e.g. $R_{500} = 8831 = P_{1100}$

**Definition 1.6** Twin Ramanujan Primes : Recall that if p and p+2 are primes, they are called twin primes.

**Twin Ramanujan Primes** are twin primes both of which are Ramanujan. They are of the form $R_n$ ; $R_{n+1}$ with $R_{n+1} = R_n + 2$ e.g. ( $R_{14}$ , $R_{15}$ ) = ( 149, 151 )

A Prime Triplet is of the form ( p, p+2, p+4 ) consisting of three primes. The set of Prime Triplets is finite .e.g. ( 3, 5, 7) is a Prime Triplet.

If $p \neq 3$, then p can be written as ( 3k , 3k+1, 3k+2 ) for some integers k = 2,3,4….

However p = 3k , then p is not prime, p = 3k +1 , then p+2 is not prime being 3(k+1)

If p= 3k+2 , then p+4 is not prime being 3(k+2)

**Bertrand's Postulate or Chebyshev theorem** :There is at least one prime p such that $x < p \leq 2x$ if $x \geq 1$

**Corollary 1.1** Any two consecutive Fibonacci numbers are relatively prime i.e. ( $F_{n+1}$ , $F_n$ ) = 1 for every n Proof : Let p be a prime factor of $F_{n+1}$ and $F_n$ Then $p / \mp 1$ which is a contradiction. Thus ( $F_{n+1}$ , $F_n$ ) =1

**Theorem 1.1** ( Cassini Formula) : $F_{n+1} F_{n-1} - F_n^2 = (-1)^n$ , for $n \geq 1$

Proof: We Prove this by induction on n

For n=1, $F_0F_1 - F_1^2 = 0.1 - 1 = -1$ , Hence it is true for n =1

Assume that it is true for n = k , $F_{k+1}. F_{k-1} - F_k^2 = (-1)^k$

To prove that it is true for n= k+1 i.e. $F_{k+2}. F_k - F^2_{k+1} = (-1)^{k+1}$

$$RHS = ( F_{k+1}. + F_k ) F_k \quad - F^2_{k+1}$$

$$= F_{k+1} F_k + F_k^2 - F_{k+1}^2$$

$$= F_{k+1} F_k + ( F_{k+1} F_{k-1} + ( -1)^{k+1} ) - F_{k+1}^2$$

$$= F_{k+1}F_k + F_{K+1} F_{k-1} - F_{k+1}^2 + ( - 1)^{k+1}$$

$$= F_{k+1} ( F_k + F_{k-1} ) - F_{k+1}^2 + ( -1 )^{k+1}$$

$$= F_{k+1}. F_{k+1} - F_{k+1}^2 + ( - 1)^{k+1}$$

$$= ( -1 )^{k+1}$$

Hence the proof.

**Divisibility of Fibonacci Numbers** :Note that a prime $p \neq 5$ divides $F_p - 1$ if and only if $p \equiv \pm 1$ ( mod ) 5 and p divides $F_p +1$ if and only if it is congruent to $\pm 2$ (mod ) 5

Note that Fibonacci primes are of the form $F_p = K_p \pm 1$ and Pell Primes $P_p = K_p \pm 1$ but Lucas Primes are not restricted Lucas numbers of prime subscripts. e.g. $L_8$ =6n -1= 47 and $L_{16}$ = 138n -1= 2207 , both of which are prime numbers.

**Theorem 1.2** The nth Ramanujan prime satisfies the inequalities $2n\log 2n < R_n < 4n\log 4n$ , $n \geq 1$

Furthermore , if $P_n$ denotes nth prime then $P_{2n} < R_n < P_{4n}$ for $n \geq 1$

**Corollary 1.2** $2p_{i-n} > p_i$ for $i > k$ where $k= \pi( p_k) = \pi ( R_n )$ **Ramanujan primes**

**Definition 1.7 Generalized Ramanujan primes** Given a constant c between 0 and 1 , the nth c-Ramanujan prime is defined as the smallest integer $R_c$ , n with the property that for any integer $x \geq R_{c,n}$ there are at least n primes between cx and x , that is , $\pi (x) - \pi (cx) \geq n$

In particular , when c= 1/2 , the nth Ramnujan prime $R_{0.5,,n} = R_n$

For c= 1/4 and 3/4 , the sequence of c-Ramanujan primes begins $R_{0.25,n}$ = 2,3,5,13,17,….

$R_{0.75,n}$ = 11,29,59,67,101,….

It is known that for all n and c , the nth c-Ramanujan prime $R_{c,n}$ exists and is prime.

**Ramanujan Prime corollary 1.3** If $P_k$ is the kth prime and $R_n$ is the nth Ramanujan prime then $2p_{i-n} > p_i$ , for $i > k$ where $k =\pi (p_k) = \pi (R_n)$

**Conclusion** :We conclude that there are various types of primes. Lucas Numbers show the differences from the Pell and Fibonacci numbers in the formation of primes. Ramanujan primes are the least integers $R_n$ for which there are at least n primes between x and x/2 for all $x \geq R_n$. Note that the integer $R_n$ is necessarily a prime number $\pi (x) - \pi ( x/2)$ and hence $\pi(x)$ must increase by obtaining another prime at $x = R_n$. Since $\pi (x) - \pi ( x/2)$ can increase by at most 1.

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