The Restrained Edge Steiner Number of A Graph

M. S. Malchijah Raj[#], J.John[*]

[#]*Research Scholar, Research and Development Centre,*
Bharathiar University, Coimbatore – 641046, India.
[*]*Department of mathematics,*
Government College of Engineering, Tirunelveli – 627007, India.

[#]malchijahraj@gmail.com
[*]john@gcetly.ac.in

***Abstract* — For a connected graph $G = (V, E)$ of order p, a set $W \subseteq V$ is called an edge Steiner set of G if every edge of G is contained in a Steiner W- tree of G. A set W of vertices of a graph G is a restrained edge Steiner set if W is an edge Steiner set, and if either $W = V$ or the subgraph $G[V - W]$ induced by $V - W$ has no isolated vertices. The minimum cardinality of a restrained edge Steiner set of G is the restrained edge Steiner number of G and is denoted by $s_{er}(G)$. The restrained edge Steiner number of certain classes of graphs is determined. Necessary conditions for the restrained edge Steiner number of a graph to be p are given. It is shown that for integers a, b and c with $3 \leq a \leq b \leq c$, there exists a connected graph G such that $s(G) = a$, $s_e(G) = b$ and $s_{er}(G) = c$.**

***Keywords*—Steiner number, edge Steiner number, restrained Steiner number, restrained edge Steiner number.**

***AMS Subject Classification*—**05C12

I. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph with no loop or multiple edge. The order and size of G are denoted by p and q respectively. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u - v$ path in G. An $u - v$ path of length d(u; v) is called an $u - v$ geodesic. For basic definitions and graph theoretic terminology we refer to [3, 1]. For a vertex v of G, the *eccentricity* $e(v)$ is the distance between v and a vertex farthest from v. The minimum eccentricity among the vertices of G is the radius, $radG$ and the maximum eccentricity is its diameter, $diamG$ of G. Two vertices x and y are *antipodal* if $d(x, y) = diamG$. A vertex v is an *extreme vertex* of a graph G if the subgraph induced by its neighbors is complete. If $e = \{u, v\}$ is an edge of a graph G with $d(u) = 1$ and $d(v) > 1$, then we call e a pendant edge, u a leaf or end vertex and v a support.

For a nonempty set W of vertices in a connected graph G, the *Steiner distance* $d(W)$ of W is the minimum size of a connected subgraph of G containing W. Necessarily, each subgraph is a tree and is called a *Steiner tree* with respect to W or a Steiner W - tree. $S(W)$ denotes the set of all vertices that lie on Steiner W - trees. A set $W \subseteq V(G)$ is called a *Steiner set* of G if every vertex of G lies on some Steiner W - tree or if $S(W) = V(G)$. A Steiner set of minimum cardinality is a *minimum Steiner set* or simply a s - set and this cardinality is the *Steiner number* $s(G)$ of G. The Steiner number of a graph was introduced in [2] and further studied in [9, 10, 11]. A set $W \subseteq V(G)$ is called an *edge Steiner set* of G if every edge of G lies on some Steiner W - tree or if $S_e(W) = E(G)$, where $S_e(W)$ denotes the set of all edges of G that lie on any Steiner W - tree. The edge Steiner number $s_e(G)$ of G is the minimum cardinality of its edge Steiner sets and any edge Steiner set of cardinality $s_e(G)$ is the minimum edge Steiner set of G or simply a s_e- set. The edge Steiner number was introduced in [10] and further studied in [7]. The restrained concept was first introduced and studied in dominating sets. The restrained concept is then applied to various parameters such as geodetic sets, edge geodetic sets, edge monophonic sets and

Steiner sets by several authors. In this paper an attempt is made to study the restrained edge Steiner number of a connected graph.

The following theorems are used in the sequel.

Theorem 1.1. [2, 10] Each extreme vertex of a connected graph G belongs to every Steiner set as well as every edge Steiner set of G.

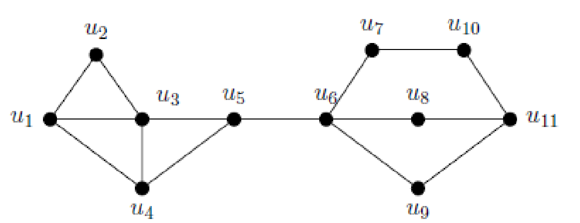
Theorem 1.2. [10] If G is a connected graph of order $p \geq 3$ such that G contains a cut vertex of degree $p-1$, then $s_e(G) = p-1$.

Theorem 1.3. [10] Let G be a connected graph of order $p \geq 3$. If G contains a vertex of degree $p-1$ and which is not a cut vertex of G, then $s_e(G) = p$.

Throughout the paper Gdenotes a connected graph with at least two vertices.

II. THE RESTRAINED EDGE STEINER NUMBER OF A GRAPH

Definition 2.1. A set W of vertices of a graph G is a restrained edge Steiner set if W is an edge Steiner set, and if either $W = V$ or the subgraph $G[V-W]$ induced by $V-W$ has no isolated vertices. The minimum cardinality of a restrained edge Steiner set of G is the restrained edge Steiner number of G, and is denoted by $s_{er}(G)$. A restrained edge Steiner set of minimum cardinality is called the s_{er} - set of G.



G
Figure 2.1

Example 2.2. For the graph G given in Figure 2.1, $W = \{u_1, u_2, u_7, u_{11}\}$ is a s - set of G and so $s(G) = 4$; $W_1 = \{u_1, u_2\, u_3, u_4, u_7, u_{11}\}$ is a s_e - set so that $s_e(G) = 6$ and $W_2 = \{u_1, u_2\, u_3, u_4, u_7, u_8, u_9, u_{10}, u_{11}\}$ is a s_{er} - set so that $s_{er}(G) = 9$. Thus the Steiner number, edge Steiner number and the restrained edge Steiner number of a graph are all different.

Theorem 2.3. For any graph G, $2 \leq s(G) \leq s_e(G) \leq s_{er}(G) \leq p$.

Proof. Any Steiner set needs at least two vertices and so $s(G) \geq 2$. Since every edge Steiner set is a Steiner set, $s(G) \leq s_e(G)$. Moreover every restrained edge Steiner set is an edge Steiner set, it follows that $s_e(G) \leq s_{er}(G)$. Also, since the vertex set $V(G)$ is a restrained edge Steiner set, it is clear that $s_{er}(G) \leq p$. ■

Remark 2.4. The bounds in Theorem 2.3 are sharp. For any non trivial path P_p $(p \geq 2)$, the set of two end vertices is the unique Steiner set so that $s(P_p) = 2$. For the cycle C_3, $s(C_3) = s_e(C_3)$. For the complete graph K_p, $s_e(K_p) = s_{er}(K_p)$. For the wheel $W_p = K_1 + C_{p-1}, (p \geq 5)$, $s_{er}(W_p) = p$. Also the inequalities in the theorem are strict. For the graph G given in Figure 2.1, $s(G) = 4$, $s_e(G) = 6$, $s_{er}(G) = 9$ and $p = 11$ so that $2 < s(G) < s_e(G) < s_{er}(G) < p$.

Since every restrained edge Steiner set of G is an edge Steiner set of G, the next result follows from Theorem 1.1.

Theorem 2.5. Each extreme vertex of a connected graph G belongs to every restrained edge Steiner set of G.

Corollary 2.6. For the complete graph $K_p(p \geq 3)$, $s_{er}(K_p) = p$.

Proof. This follows from Theorem 2.5. ∎

Theorem 2.7. Let G be a connected graph with v a cut vertex of G and let W be a restrained edge Steiner set of G. Then every component of $G - v$ contains an element of W.

Proof. Let v be a cut vertex of G and W be a restrained edge Steiner set of G. Suppose there exists a component say G_1 of $G - v$ such that G_1contains no vertex of W. By Theorem 2.5, W contains all the extreme vertices of G and hence it follows that G_1 does not contain any extreme vertex of G. ThusG_1contains at least one edge say xy. Since every Steiner W - tree T must have its end-vertex in W and v is a cut vertex of G, it is clear that no Steiner W - tree would contain the edge xy. This contradicts that W is a restrained edge Steiner set of G. ∎

Corollary 2.8. If v is a cut vertex of a connected graph G and W is a restrained edge Steiner set of G, then v lies in every Steiner W - tree of G

In [7], it is proved that, if v is a cut vertex of a connected graph G and $W \subseteq V(G)$ with $v \notin W$, then $W \cup \{v\}$ is an edge Steiner set of G if and only if W is an edge Steiner set of G. But the result need not be true for the restrained edge Steiner number of a graph.

Remark 2.9. For the graph G given in Figure 2.2, v_2 is a cut vertex and $W = \{v_1, v_4, v_6\}$ is the restrained edge Steiner set. However $W \cup \{v_2\}$ is not a restrained edge Steiner set of G.

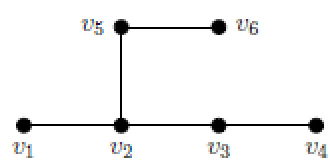


Figure 2.2

Theorem 2.10. There is no graph of order p with $s_{er}(G) = p - 1$.

Proof. Since every restrained edge Steiner set of G is an edge Steiner set of G and the complement of each restrained edge Steiner set has cardinality different from one, we have $s_{er}(G) \neq p - 1$. ∎

Theorem 2.11. Let G be a non trivial tree which is not a star. Then $s_{er}(G)$ is equal to the set of all end vertices of G.

Proof. Let W be the set of all end vertices of G. Then W ia a restrained edge Steiner set of G. By Theorem 2.5, $s_{er}(G) \leq |W|$. Since $G \neq K_{1,p-1}$, the subgraph $G[V - W]$ has no isolated vertices. Therefore W is a restrained edge Steiner set of G so that $s_{er}(G) = |W|$. ∎

Corollary 2.12. For any tree T with $p \geq 3$ vertices, $s_{er}(T) = p$ if and only if T is a star.

Proof. This follows from Theorem 2.11 and 2.10. ∎

Corollary 2.13. For any tree T with $p \geq 4$ vertices, $s_{er}(T) = p - 2$ if and only if T is a double star.

Proof. This follows from Theorem 2.11. ∎

Theorem 2.14. For the cycle $G = C_p (p \geq 3)$, $s_{er}(G) = \begin{cases} p \text{ for } p \in \{3,4,5\}; \\ 2 \text{ for } p \geq 6 \text{ and } p \text{ is even}; \\ 3, \text{for } p \geq 7 \text{ and } p \text{ is odd}. \end{cases}$

Proof. It is clear that $s_{er}(Cp) = p$ for $p \in \{3, 4, 5\}$. Let $p \geq 6$. Let p be even and $W = \{u, v\}$ be a set of antipodal vertices of Cp. Then W is an edge Steiner set of G. Since the subgraph $G[V - W]$ has no isolated vertices, W is a restrained edge Steiner set of G so that $s_{er}(Cp) = 2$. Let p be odd. It is clearly verified that no two element subset of Cp is an edge Steiner set of G and so $s_{er}(G) \geq 3$. For any vertex u, let v and w be the two antipodal vertices. Let $W_1 = \{u, v, w\}$. Then W_1 is an edge Steiner set of G. Since $p \geq 7$, the subgraph $G[V - W_1]$ has no isolated vertices. Then W_1 is a restrained edge Steiner set of G so that $s_{er}(Cp) = 3$. ∎

Theorem 2.15. For the complete bipartite graph $K_{m,n}, (m, n \geq 2)$, $s_{er}(K_{m,n}) = m + n$.

Proof. Let $G = K_{m,n}, (2 \leq m \leq n)$. Let X and Y be the bipartite sets of G, where $X = \{x_1, x_2, \dots, x_m\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. Let $2 \leq m < n$. By Theorem 1.2, $W = X$ is a s - set of G. Since the subgraph $G[V - W]$ has isolated vertices, W is not a restrained edge Steiner set of G and so $s_{er}(G) \geq m + 1$. Let W' be a restrained edge Steiner set of G with $|W'| \geq m + 1$. Then either $W' \subseteq Y$ or $W' \subsetneq X \cup Y$. If $W' \subseteq Y$, then $W' = Y$ is the only edge Steiner set of G. Since the subgraph $G[V - W']$ has isolated vertices, $W'$is not a restrained edge Steiner set of G, which is a contradiction. If $W' \subsetneq X \cup Y$, then $\langle W' \rangle$ is connected. Then the Steiner W'- tree contains elements ofW'only. Therefore W' is not a restrained edge Steiner set of G. Hence $W' = X \cup Y$. This implies $s_{er}(G) = m + n$. Similarly if $m = n$, we can prove that $s_{er}(G) = m + n$. ∎

Theorem 2.16. For the hypercube $Q_n (n \geq 3), s_{er}(Q_n) = 2$.

Proof. Q_n has 2^n vertices, which may be labeled $a_1 a_2 a_3 \dots\dots a_n$, where each $a_i (1 \leq i \leq n)$ is either 0 or 1. It is easily seen that $\{(0, 0, 0, \dots\dots, 0), (1, 1, 1, \dots\dots, 1)\}$ is a $s_{er}(Q_n)$ set for $n \geq 3$. Hence $s_{er}(Q_n) = 2$. ∎

Theorem 2.17. Let G be a connected graph with v a vertex of degree $p - 1$. Then $s_{er}(G) = p$.

Proof. Let v be the vertex of degree $p - 1$ in G.
Case 1. Suppose that v is a cut vertex of G. Then by Theorem 1.2, $s_e(G) = p - 1$. Hence by Theorem 2.3, $s_{er}(G) = p - 1$ or p. By Theorem 2.10, $s_{er}(G) \neq p - 1$. Hence $s_{er}(G) = p$.

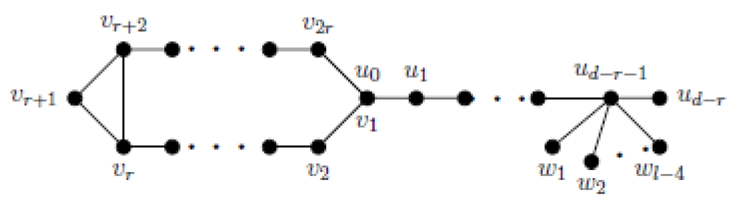
Case 2. Suppose that v is not a cut vertex of G. Then by Theorem 1.3, $s_e(G) = p$. Now by Theorem 2.3, $s_{er}(G) = p$. ∎

Theorem 2.18. For the wheel $W_p = K_1 + C_{p-1}, (p \geq 5),\ s_{er}(W_p) = p$.

Proof. The proof follows from Theorem 2.17. ■

Theorem 2.19. For positive integers r, d and $l \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with $radG = r$, $diamG = d$, $s_e(G) = l$ and $s_{er}(G) = l$.

Proof. When $r = 1$, then $d = 1$ or $d = 2$. If $d = 1$, then $G = K_l$ has the desired properties. If $d = 2$, then $G = K_{1,l-1}$ has the desired properties. If $r = d = 2$, then $G = K_{2,l-2}$ has the desired properties. Now, let $r \geq 3$. Construct a graph G with the desired properties as follows: Let $C_{2r}: v_1, v_2, \dots, v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1}: u_0, u_1, u_2, \dots, u_{d-r}$ be a path of order $d - r + 1$ and length $d - r$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Then add $l - 4$ new vertices $w_1, w_2, \dots, w_{l-4}$ to H and join each vertex $w_i (1 \leq i \leq l-4)$ to the vertex u_{d-r-1} and obtain the graph G of Figure 2.3.

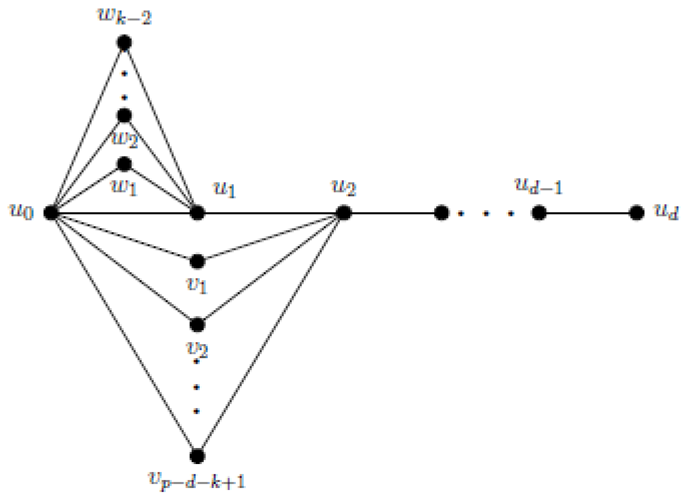


G
Figure 2.3

Then $radG = r$ and $diamG = d$. Let $W = \{w_1, w_2, \dots, w_{l-4}, u_{d-r}\}$ be the set of all end vertices of G. By Theorem 1.1 and Theorem 2.5 each extreme vertex of G belongs to every edge Steiner set as well as every restrained edge Steiner set of G. Let $W = \{w_1, w_2, \dots, w_{l-4}, u_{d-r}\}$. Since no vertex or no edge on the cycle C_{2r} lies on a Steiner W - tree of G, we see that W is not an edge Steiner set of G. Let $W' = W \cup \{v_{r+1}\}$, where v_{r+1} is the antipodal vertex of u_{d-r} in G. Then W' is a Steiner set of G. Since the edge $v_r v_{r+2}$ does not lie on the Steiner W' tree, W' is not an edge Steiner set of G. Let $W'' = W' \cup \{v_r, v_{r+2}\}$. Then it is clear that $W''$is an edge steiner set of G. Since the subgraph $G[V - W'']$ has no isolated vertices, W'' is a restrained edge Steiner set of G so that $s_e(G) = l$ and $s_{er}(G) = l$. ■

Theorem 2.20. If p, d and k are integers such that $2 \leq d \leq p$, $2 \leq k < p$ and $p - d - k + 1 > 0$, then there exists a graph G of order p, diameter d and $s_e(G) = s_{er}(G) = k$.

Proof. Let $P_d: u_0, u_1, \dots, u_d$ be a path of length d. Add new vertices $v_1, v_2, \dots, v_{p-d-k+1}$ and $w_1, w_2, \dots, w_{k-2}$ and join each v_i $(1 \leq i \leq p - d - k + 1)$ with u_0 and u_2 and also join each w_i $(1 \leq i \leq k-2)$ with u_0 and u_1 in P_d, there by obtaining the graph G in Figure 2.4. Let $W = \{w_1, w_2, \dots, w_{k-2}, u_d\}$ be the set of all extreme vertices of G with $|W| = k - 1$. By Theorems 1.1 and 2.5, W is a subset of every edge Steiner set as well as every restrained edge Steiner set of G. It is clear that W is neither an edge Steiner set nor a restrained edge Steiner set of G. Now it is easily observed that $W \cup \{u_0\}$ is an edge Steiner set as well as a restrained edge Steiner set of G so that $s_e(G) = s_{er}(G) = k$. ■



G
Figure 2.4

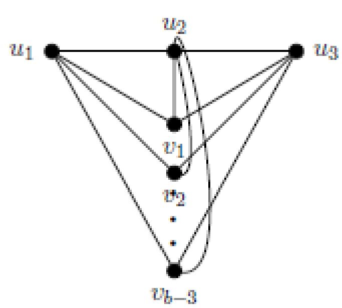
III. Realization Result

In view of Theorem 2.3, we give a realization result for the Steiner number, the edge Steiner number and the restrained edge Steiner number of a graph.

Theorem 3.1. For any positive integers a, b and c with $2 \leq a \leq b \leq c$ and $b - a - 1 \neq 0$ and $c - b - 1 \neq 0$, there exists a connected graph G such that $s(G) = a, s_e(G) = b$ and $s_{er}(G) = c$.

Proof. We consider 4 cases.

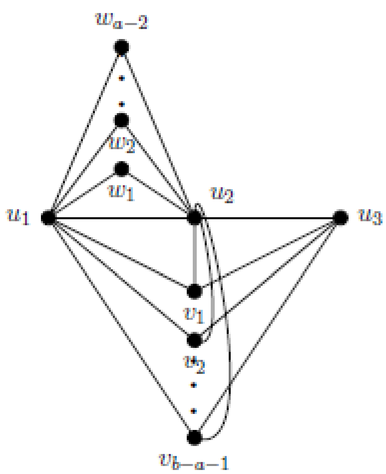
Case 1. $a = 2, b = c$. Let G be the graph in Figure 3.1 obtained from the path on three vertices P u_1, u_2, u_3 by adding $b - 3$ new vertices $v_1, v_2, \ldots, v_{b-3}$ and joining each $v_i (1 \leq i \leq b - 3)$ with u_1, u_2 and u_3. It is clear that $\{u_1, u_3\}$ is a Steiner set of G so that $s(G) = 2 = a$. Since u_2 is a full degree vertex of G, it follows from Theorems 1.4 and 2.17 that $s_e(G) = s_{er}(G) = b - 3 + 3 = b$.



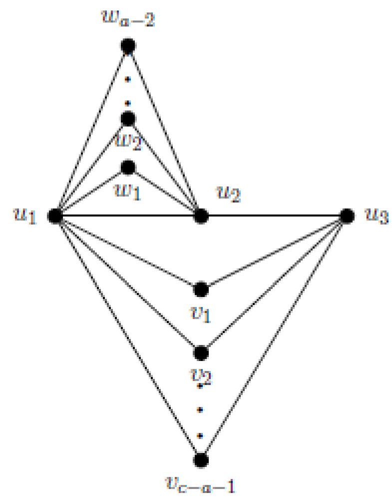
G
Figure 3.1

Case 2. $3 \leq a < b = c$. Let G be the graph in Figure 3.2 obtained from the path on three vertices P: u_1, u_2, u_3 by adding the new vertices $v_1, v_2, \ldots, v_{b-a-1}$ and $w_1, w_2, \ldots, w_{a-2}$ and joining each $v_i (1 \leq i \leq b - a - 1)$ with u_1, u_2 and u_3 and also joining each $w_i (1 \leq i \leq a - 2)$ with u_1 and u_2. Let $W = \{w_1, w_2, \ldots, w_{a-2}\}$ be the set of all extreme vertices of G. Let S be any Steiner set of G. Then by Theorem

1.1, $W \subseteq S$. It is clear that W is not a Steiner set of G. Also it is easily observed that $W \cup \{v\}$ where $v \notin W$ is not a Steiner set of G. It is clear that $W \cup \{u_2, u_3\}$ is a Steiner set of G and so $s(G) = a$. Since u_2 is a full degree vertex of G, it follows from Theorems 1.4 and 2.17 that $s_e(G) = s_{er}(G) = b - 3 + 3 = b$.



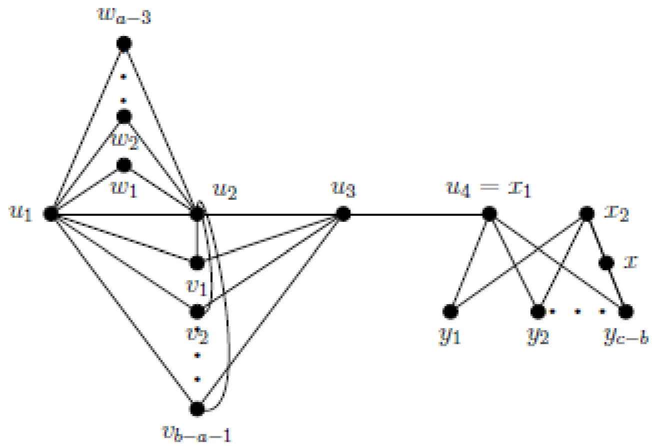
G
Figure 3.2



G
Figure 3.3

Case 3. $3 \leq a = b < c$. Let G be the graph shown in Figure 3.3, obtained from the path P on three vertices $u_1,\ u_2$ and u_3 by adding the new vertices $v_1,\ v_2, \ldots,\ v_{c-a-1}$ and $w_1,\ w_2, \ldots,\ w_{a-2}$ and joining each $v_i(1 \leq i \leq c-a-1)$ with u_1 and u_3 and also joining each $w_i(1 \leq i \leq a-2)$ with u_1 and u_2. Let $W = \{w_1, w_2, \ldots, w_{a-2}\}$ be the set of all extreme vertices of G. By Theorem 1.1, W is a subset of every Steiner set of G. Since $S(W) \neq V$, W is not a Steiner set of G. Hence $s(G) \geq a-2$. It is easily verified that $W \cup \{u\}$, where $u \notin W$, is not a Steiner set of G and so $s(G) \geq a-1$. It is clear that $W_1 = W \cup \{u_1, u_3\}$ is a Steiner set of G so that $s(G) = a$. By the similar argument we can prove that $s_e(G) = a$. Since the subgraph $G[V - W_1]$ contains isolated vertices, W_1 is not a restrained edge Steiner set of G. By Theorem 2.5, W is a subset of every restrained edge Steiner set of G. It is easily seen that every restrained edge Steiner set contains each $v_i(1 \leq i \leq c-a-1)$. Let $W_2 = W_1 \cup \{u_2, v_1,\ v_2, \ldots,\ v_{c-a-1}\}$. Then W_2 is a restrained edge Steiner set of G and so $s_{er}(G) = a + c - a = c$.

Case 4. $3 \leq a < b < c$. Let $G_1 = K_{2,c-b}$ be the complete bipartite graph with partite sets $X = \{x_1, x_2\}$ and $Y = \{y_1, y_2, \ldots, y_{c-b}\}$ and let $P: u_1,\ u_2,\ u_3, u_4$ be a path of order 4. Let H be the graph obtained by adding new vertices $v_1,\ v_2, \ldots,\ v_{b-a-1}$ and $w_1,\ w_2, \ldots,\ w_{a-3}$ and join each $v_i(1 \leq i \leq b-a-1)$ with $u_1,\ u_2$ and u_3 and also join each $w_i(1 \leq i \leq a-3)$ with u_1 and u_2 in P. Let G be the graph obtained from G_1 and H by identifying the vertex x_1 in G_1 with the vertex u_4 in H and subdividing the edge x_2y_{c-b} of G_1 by the vertex x. The graph G is shown in Figure 3.4.



G
Figure 3.4

Since each $w_i (1 \leq i \leq a-3)$ is an extreme vertex of G, by Theorem 1.1, each $w_i (1 \leq i \leq a-3)$ belongs to every Steiner set of G. Let $W = \{w_1, w_2, \ldots, w_{a-3}\}$. Then W is not a Steiner set of G and so $s(G) \geq a-2$. Also it is easily seen that neither $W \cup \{u\}$ where $u \notin W$ nor $W \cup \{u, v\}$ where $u, v \notin W$ is not a Steiner set of G. Now it is easily verified that $W_1 = W \cup \{u_1, x_2, y_{c-b}\}$ is a Steiner set of G so that $s(G) = a$. Next we show that $s_e(G) = b$. Since the edges $u_2v_1, u_2v_2, \ldots, u_2v_{b-a-1}$ do not lie on any Steiner W_1- tree, W_1 is not an edge Steiner set of G. By Theorem 1.1, W is a subset of every edge Steiner set of G. Also it is easily seen that the vertices u_2 and each $v_i (1 \leq i \leq b-a-1)$ belong to every edge Steiner set of G and so $s_e(G) \geq b-a$. Let $W_2 = W_1 \cup \{u_2, v_1, v_2, \ldots, v_{b-a-1}\}$. Then W_2 is an edge Steiner set of G so that $s_e(G) = b$. Since the subgraph $G[V - W_2]$ contains an isolated vertex, W_2 is not a restrained edge Steiner set of G. By Theorem 2.5, W is a subset of every restrained edge Steiner set of G. Let $V = \{v_1, v_2, \ldots, v_{b-a-1}\}$. Let $W' = W \cup Y \cup V \uplus \{u_1, u_2\}$. It is easily seen that W' is not a restrained edge Steiner set of G and so $s_{er}(G) \geq c-1$. It is easily observed that $W' \cup \{v\}$ where $v \notin W'$ is not a restrained edge Steiner set of G and so $s_{er}(G) \geq c$. However $W_3 = W' \cup \{x, x_2\}$ is a restrained edge Steiner set of G and so $s_{er}(G) = c$. ■

REFERENCES

[1] F. Buckley and F. Harary, Distance in Graphs, *Addition-Wesley, Redwood City, CA*, 1990.
[2] G. Chartrand, P. Zhang, The Steiner Number of a Graph, *Discrete Math.* **242**(1) (2002), 41 –54.
[3] F. Harary, Graph Theory, *Addition-Wesley*, 1969.
[4] J. John, M. S. Malchijah Raj, The Restrained Steiner Number of a Graph (Communicated).
[5] M. S. Malchijah Raj, J. John, The Upper Restrained Steiner Number of a Graph (Accepted).
[6] M. S. Malchijah Raj, J. John, The Forcing Restrained Steiner Number of a Graph (Communicated).
[7] Michael B. Frondoza and Sergio R. Canoy, Jr., The Edge Steiner Number of a Graph, *Bulletin of the Malays. Math. Sci. Soc.* (2) **35** (1) (2012), 53 - 59.
[8] P. A. Ostrand, Graphs with Specified radius and diameter, *Discrete Math.* **4** (1973) 71 – 75.
[9] I. M. Paleyo, Comment on "The Steiner number of a graph" by G. Chartrand and P. Zhang, *Discrete Math*. **242** (2002), 41 – 54.
[10] A. P. Santhakumaran and J. John, The Edge Steiner number of a graph, *Journal of Discrete Mathematical Sciences and Cryptography*, **10** (2007) 677 – 696.
[11] A. P. Santhakumaran and J. John, The Forcing Steiner number of a graph, *Discussiones Mathematicae Graph Theory*, **31** (2011) 171 – 181.